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Holography beyond the Penrose limit

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Abstract

The flat pp-wave background geometry has been realized as a particular Penrose limit of $AdS_5 \times S^5$. It describes a string that has been infinitely boosted along an equatorial null geodesic in the S^5 subspace. The string worldsheet Hamiltonian in this background is free. Finite boosts lead to curvature corrections that induce interacting perturbations of the string worldsheet Hamiltonian. We develop a systematic light-cone gauge quantization of the interacting worldsheet string theory and use it to obtain the interacting spectrum of the so-called 'two-impurity' states of the string. The quantization is technically rather intricate and we provide a detailed account of the methods we use to extract explicit results. We give a systematic treatment of the fermionic states and are able to show that the spectrum possesses the proper extended supermultiplet structure (a non-trivial fact since half the supersymmetry is nonlinearly realized). We test holography by comparing the string energy spectrum with the scaling dimensions of corresponding gauge theory operators. We confirm earlier results that agreement obtains in low orders of perturbation theory, but breaks down at third order. The methods presented here can be used to explore these issues in a wider context than is specifically dealt with in this paper.

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1 Introduction

The AdS/CFT correspondence encompasses a wide range of holographic mappings between string theory and gauge theory. In its original incarnation, Maldacena's conjecture states that type IIB superstring theory in $AdS_5 \times S^5$ with N_c units of Ramond-Ramond (RR) flux on the sphere is equivalent to $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory in $3+1$ dimensions [1]. This correspondence is conjectured to be precise for the identification $g_s = g_{YM}^2$. The holographically dual gauge theory is defined on the conformal boundary of $AdS_5 \times S^5$, or $\mathbf{R} \times S^3$. Strong evidence in support of this particular correspondence is the fact that both sides of the duality have the same symmetry, $PSU(2,2|4)$. While a great deal of additional evidence in support of the conjecture has accumulated in recent years, a direct verification of Maldacena's original proposal has been elusive. It is difficult to quantize the superstring theory using the RNS formalism in the presence of background RR fields. The GS formalism naturally accommodates RR backgrounds but, despite the high degree of symmetry of the $AdS_5 \times S^5$ background, a gauge choice leading to a solvable theory has not been found. Early studies of the duality have therefore concentrated on the supergravity approximation to the string theory (and, of course, yield many impressive results).

In order to address specifically stringy aspects of the duality, it has been necessary to consider simplifying limits of the canonical $AdS_5 \times S^5$ background. Metsaev [2] showed that, in a certain plane-wave geometry supported by a constant RR flux, light-cone gauge worldsheet string theory reduces to a free theory with the novel feature that the worldsheet bosons and fermions acquire a mass. This solution was later shown to be a Penrose limit of the familiar $AdS_5 \times S^5$ supergravity solution [3], and describes the geometry near a null geodesic boosted around the equator of the S^5 subspace. The energies of Metsaev's free string theory are thus understood to be those of a string in the full $AdS_5 \times S^5$ space, in the limit that the states are boosted to large angular momentum about an equatorial circle in the S^5. Callan, Lee, McLoughlin, Schwarz, Swanson and Wu [4] subsequently calculated the corrections (in inverse powers of the angular momentum) to the string spectrum that arise if the string is given a large, but finite, boost. Comparison of the resulting interacting spectrum with corrections (in inverse powers of $\mathcal{R}$-charge) to the dimensions of the corresponding gauge theory operators largely (but not completely) confirms expectations from AdS/CFT duality (see [4, 5] for discussion). The purpose of this work is not to present new results, but rather to describe in fairly complete detail the methods used to obtain the results presented in [4] (but only outlined in that paper). Some aspects of the purely bosonic side of this problem were studied by Parnachev and Ryzhov [6]. Although we find no disagreement with them, our approach differs from theirs in certain respects, most notably in taking full account of supersymmetry.

Our approach is to take the GS superstring action on $AdS_5 \times S^5$, constructed using the formalism of Cartan forms and superconnections on the $SU(2,2|4)/(SO(4,1) \times SO(5))$ coset superspace [7], expand it in powers of the background curvature and finally eliminate unphysical degrees of freedom by light-cone gauge quantization. We treat the resulting interaction Hamiltonian in first-order degenerate perturbation theory to find the first corrections

to the highly-degenerate pp-wave spectrum. The complexity of the problem is such that we are forced to resort to symbolic manipulation programs to construct and diagonalize the perturbation matrix. In this paper we give a proof of principle by applying our methods to the subspace of two-impurity excitations of the string. We show that the spectrum organizes itself into correct extended supersymmetry multiplets whose energies match well (if not perfectly) with what is known about gauge theory anomalous dimensions.

In section 2 we introduce the problem by considering the bosonic sector of the theory alone. We comment on some interesting aspects of the theory that arise when restricting to the point-particle (or zero-mode) subsector. In section 3 we review the construction of the GS superstring action on $AdS_5 \times S^5$ as a nonlinear sigma model on the $SU(2,2|4)/(SO(4,1) \times SO(5))$ coset superspace. In sections 4 and 5 we perform a large-radius expansion on the relevant objects in the theory, and carry out the light-cone gauge reduction, thereby extracting explicit curvature corrections to the pp-wave Hamiltonian. Section 6 presents results on the curvature-corrected energy spectrum, further expanded to linear order in the modified 't Hooft coupling $\lambda' = g_{YM}^2 N_c/J^2$; results from corresponding gauge theory calculations (at one loop in $\lambda = g_{YM}^2 N_c$) are summarized and compared with the string theory. In section 7 we extend the string theory analysis to higher orders in λ', and compare results with what is known about gauge theory operator dimensions at higher-loop order. The final section is devoted to discussion and conclusions.

2 Strings beyond the Penrose limit: General considerations

To introduce the computation of finite-J corrections to the pp-wave string spectrum, we begin by discussing the construction of the light-cone gauge worldsheet Hamiltonian for the bosonic string in the full $AdS_5 \times S^5$ background. The problem is much more complicated when fermions are introduced, and we will take up that aspect of the calculation in a later section. A study of the purely bosonic problem gives us the opportunity to explain various strategic points in a simpler context.

In convenient global coordinates, the $AdS_5 \times S^5$ metric can be written in the form

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2) \ , \tag{2.1}$$

where R denotes the radius of both the sphere and the AdS space, and $d\Omega_3^2$, $d\tilde{\Omega}_3^2$ denote separate three-spheres. The coordinate ϕ is periodic with period 2π and, strictly speaking, so is the time coordinate t. In order to accommodate string dynamics, it is necessary to pass to the covering space in which time is *not* taken to be periodic. This geometry, supplemented by an RR field with N_c units of flux on the sphere, is a consistent, maximally supersymmetric type IIB superstring background, provided that $R^4 = g_s N_c (\alpha')^2$ (where g_s is the string coupling).

In its initial stages, development of the AdS/CFT correspondence focused on the supergravity approximation to string theory in $AdS^5 \times S^5$. Recently, attention has turned to the

problem of evaluating truly stringy physics in this background and studying its match to gauge theory physics. The obstacles to such a program, of course, are the general difficulty of quantizing strings in curved geometries, and the particular problem of defining the superstring in the presence of RR background fields. As noted above, the string quantization problem is partly solved by looking at the dynamics of a string that has been boosted to lightlike momentum along some direction, or, equivalently, by quantizing the string in the background obtained by taking the Penrose limit of the original geometry using the lightlike geodesic corresponding to the boosted trajectory. The simplest choice is to boost along the equator of the S^5 or, equivalently, to take the Penrose limit with respect to the lightlike geodesic $\phi = t,\ \rho = \theta = 0$ and to quantize the system in the appropriate light-cone gauge.

To quantize about the lightlike geodesic at $\rho = \theta = 0$, it is helpful to make the reparameterizations

$$\cosh\rho = \frac{1+z^2/4}{1-z^2/4} \qquad \cos\theta = \frac{1-y^2/4}{1+y^2/4}\ , \tag{2.2}$$

and work with the metric

$$ds^2 = R^2\left[-\left(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{dy_{k'}dy_{k'}}{(1+\frac{1}{4}y^2)^2}\right]\ . \tag{2.3}$$

The $SO(8)$ vectors spanning the eight directions transverse to the geodesic are broken into two $SO(4)$ subgroups parameterized by $z^2 = z_k z^k$ with $k = 1,\ldots,4$, and $y^2 = y_{k'}y^{k'}$ with $k' = 5,\ldots,8$. This form of the metric is well-suited for the present calculation: the spin connection, which will be important for the superstring action, turns out to have a simple functional form and the AdS_5 and S^5 subspaces appear nearly symmetrically. This metric has the full $SO(4,2)\times SO(6)$ symmetry associated with $AdS_5\times S^5$, but only the translation symmetries in t and ϕ and the $SO(4)\times SO(4)$ symmetry of the transverse coordinates remain manifest. The translation symmetries mean that string states have a conserved energy ω, conjugate to t, and a conserved (integer) angular momentum J, conjugate to ϕ. Boosting along the equatorial geodesic is equivalent to studying states with large J, and the lightcone Hamiltonian gives eigenvalues for $\omega - J$ in that limit. On the gauge theory side, the S^5 geometry is replaced by an $SO(6)$ $\mathcal{R}$-symmetry, and J corresponds to the eigenvalue of an $SO(2)$ $\mathcal{R}$-symmetry generator. The AdS/CFT correspondence implies that string energies in the boosted limit should match operator dimensions in the limit of large $\mathcal{R}$-charge (a limit in which perturbative evaluation of operator dimensions becomes legitimate).

On dimensional grounds, taking the $J\to\infty$ limit on the string states is equivalent to taking the $R\to\infty$ limit on the metric (in the right coordinates). The coordinate redefinitions

$$t\to x^+ - \frac{x^-}{2R^2} \qquad \phi\to x^+ + \frac{x^-}{2R^2} \qquad z_k\to\frac{z_k}{R} \qquad y_{k'}\to\frac{y_{k'}}{R} \tag{2.4}$$

make it possible to take a smooth $R\to\infty$ limit. Expressing the metric (2.3) in these new

coordinates, we obtain the following expansion in powers of $1/R^2$:

$$
\begin{aligned}
ds^2 \;\approx\;& 2\,dx^+dx^- + dz^2 + dy^2 - \left(z^2+y^2\right)(dx^+)^2 + \\
& \left[2\left(z^2-y^2\right)dx^-dx^+ + z^2dz^2 - y^2dy^2 - \left(z^4-y^4\right)(dx^+)^2\right]\frac{1}{2R^2} \\
& +\mathcal{O}(1/R^4)\ . \qquad (2.5)
\end{aligned}
$$

The leading R-independent part is the well-known pp-wave metric. The coordinate x^+ is dimensionless, x^- has dimensions of length squared, and the transverse coordinates now have dimensions of length. Since it is quadratic in the eight transverse bosonic coordinates, the pp-wave limit leads to a quadratic (and hence soluble) Hamiltonian for the bosonic string. The $1/R^2$ corrections to the metric are what will eventually concern us: they will add quartic interactions to the light-cone Hamiltonian and lead to first-order shifts in the energy spectrum of the string.

After introducing lightcone coordinates $x^\pm$ according to (2.4), the general $AdS_5\times S^5$ metric can be cast in the form

$$
ds^2 = 2G_{+-}dx^+dx^- + G_{++}dx^+dx^+ + G_{--}dx^-dx^- + G_{AB}dx^Adx^B\ , \qquad (2.6)
$$

where x^A ($A=1,\dots,8$) labels the eight transverse directions, the metric components are functions of the x^A only, and the components G_{+A} and G_{-A} are not present. This simplifies even further for the pp-wave metric, where $G_{--}=0$ and $G_{+-}=1$. We will use (2.6) as the starting point for constructing the light-cone gauge worldsheet Hamiltonian (as a function of the transverse x^A and their conjugate momenta p_A) and for discussing its expansion about the free pp-wave Hamiltonian.

The general bosonic Lagrangian density has a simple expression in terms of the target space metric:

$$
\mathcal{L} = \frac{1}{2}h^{ab}G_{\mu\nu}\partial_a x^\mu\partial_b x^\nu\ , \qquad (2.7)
$$

where h is built out of the worldsheet metric γ according to $h^{ab}=\sqrt{-\det\gamma}\,\gamma^{ab}$ and the indices a,b label the worldsheet coordinates σ,τ. Since $\det h=-1$, there are only two independent components of h. The canonical momenta (and their inversion in terms of velocities) are

$$
p_\mu = h^{\tau a}G_{\mu\nu}\partial_a x^\nu\ , \qquad \dot{x}^\mu = \frac{1}{h^{\tau\tau}}G^{\mu\nu}p_\nu - \frac{h^{\tau\sigma}}{h^{\tau\tau}}x'^\mu\ . \qquad (2.8)
$$

The Hamiltonian density $\mathcal{H}=p_\mu\dot{x}^\mu-\mathcal{L}$ is

$$
\mathcal{H} = \frac{1}{2h^{\tau\tau}}(p_\mu G^{\mu\nu}p_\nu + x'^\mu G_{\mu\nu}x'^\nu) - \frac{h^{\tau\sigma}}{h^{\tau\tau}}(x'^\mu p_\mu)\ . \qquad (2.9)
$$

As is usual in theories with general coordinate invariance (on the worldsheet in this case), the Hamiltonian is a sum of constraints times Lagrange multipliers built out of metric coefficients ($1/h^{\tau\tau}$ and $h^{\tau\sigma}/h^{\tau\tau}$).

One can think of the dynamical system we wish to solve as being defined by $\mathcal{L} = p_\mu \dot{x}^\mu - \mathcal{H}$ (a phase space Lagrangian) regarded as a function of the coordinates x^μ, the momenta p_μ and the components h^{ab} of the worldsheet metric. To compute the quantum path integral, the exponential of the action constructed from this Lagrangian is functionally integrated over each of these variables. For a spacetime geometry like (2.6), one finds that with a suitable gauge choice for the worldsheet coordinates (τ, σ), the functional integrations over all but the transverse (physical) coordinates and momenta can be performed, leaving an effective path integral for these physical variables. This is the essence of the light-cone approach to quantization.

The first step is to eliminate integrations over x^+ and p_- by imposing the light-cone gauge conditions $x^+ = \tau$ and $p_- = \text{const}$. (At this level of analysis, which is essentially classical, we will not be concerned with ghost determinants arising from this gauge choice.) As noted above, integrations over the worldsheet metric cause the coefficients $1/h^{\tau\tau}$ and $h^{\tau\sigma}/h^{\tau\tau}$ to act as Lagrange multipliers, generating delta functions that impose two constraints:

$$\begin{gathered} x'^- p_- + x'^A p_A = 0 \\ G^{++} p_+^2 + 2G^{+-} p_+ p_- + G^{--} p_-^2 + p_A G^{AB} p_B + x'^A G_{AB} x'^B + G_{--} \frac{(x'^A p_A)^2}{p_-^2} = 0 \ . \end{gathered} \tag{2.10}$$

When integrations over x^- and p_+ are performed, the constraint delta functions serve to evaluate x^- and p_+ in terms of the dynamical transverse variables (and the constant p_-). The first constraint is linear in x^- and yields $x'^- = -x'^A p_A / p_-$. Integrating this over σ and using the periodicity of x^- yields the standard level-matching constraint, without any modifications. The second constraint is quadratic in p_+ and can be solved explicitly for $p_+ = -\mathcal{H}_{lc}(x^A, p_A)$. The remaining transverse coordinates and momenta have dynamics which follow from the phase space Lagrangian

$$\mathcal{L}_{\text{ps}} = p_+ + p_- \dot{x}^- + p_A \dot{x}^A \sim p_A \dot{x}^A - \mathcal{H}_{lc}(x^A, p_A) \ , \tag{2.11}$$

where we have eliminated the p_- term by integrating by parts in time and imposing that p_- is constant. The essential result is that $-p_+ = \mathcal{H}_{lc}$ is the Hamiltonian that generates evolution of the physical variables x^A, p_A in worldsheet time τ. This is, of course, dynamically consistent with the light-cone gauge identification $x^+ = \tau$ (which requires worldsheet and target space time translation to be the same).

We can solve the quadratic constraint equation (2.10) for $p_+ = -\mathcal{H}_{\text{lc}}$ explicitly, obtaining the uninspiring result

$$\mathcal{H}_{\text{lc}} = -\frac{p_- G_{+-}}{G_{--}} - \frac{p_- \sqrt{G}}{G_{--}} \sqrt{1 + \frac{G_{--}}{p_-^2}(p_A G^{AB} p_B + x'^A G_{AB} x'^B) + \frac{G_{--}^2}{p_-^4}(x'^A p_A)^2} \ , \tag{2.12}$$

where

$$G \equiv G_{+-}^2 - G_{++} G_{--} \ . \tag{2.13}$$

This is not very useful as it stands, but we can put it in more manageable form by expanding it in powers of $1/R^2$. We can actually do slightly better by observing that the constraint

equation (2.10) becomes a linear equation for p_+ if $G_{--} = 0$ (which is equivalent to $G^{++} = 0$). Solving the linear equation for p_+ gives

$$\mathcal{H}_{\rm lc} = \frac{p_- G_{++}}{2G_{+-}} + \frac{G_{+-}}{2p_-}(p_A G^{AB} p_B + x'^A G_{AB} x'^B) \ , \tag{2.14}$$

a respectable non-linear sigma model Hamiltonian. In the general $AdS_5 \times S^5$ metric (2.1) we cannot a convenient set of coordinates such that G_{--} identically vanishes. Using (2.4), however, we can find coordinates where G_{--} has an expansion which begins at $\mathcal{O}(1/R^4)$, while the other metric coefficients have terms of all orders in $1/R^2$. Therefore, if we expand in $1/R^2$ and keep terms of at most $\mathcal{O}(1/R^2)$, we may set $G_{--} = 0$ and use (2.14) to construct the expansion of the lightcone Hamiltonian to that order. The leading $\mathcal{O}(R^0)$ terms in the metric reproduce (as they should) the bosonic pp-wave Hamiltonian

$$\mathcal{H}_{\rm lc}^{pp} \ = \ \frac{1}{2}\left[(\dot{p}^A)^2 + (x'^A)^2 + (x^A)^2\right] \ , \tag{2.15}$$

(choosing $p_- = 1$ for the conserved worldsheet momentum density). The $\mathcal{O}(1/R^2)$ terms generate a perturbing Hamiltonian density which is quartic in fields and quadratic in worldsheet time and space derivatives:

$$\mathcal{H}_{\rm lc}^{R^{-2}} = \frac{1}{4R^2}(y^2 p_z^2 - z^2 p_y^2) + \frac{1}{4R^2}((2z^2 - y^2)(z')^2 - (2y^2 - z^2)(y')^2) \ . \tag{2.16}$$

This is the bosonic part of the perturbing Hamiltonian we wish to derive. If we express it in terms of the creation and annihilation operators of the leading quadratic Hamiltonian (2.15) we can see that its matrix elements will be of order $1/J$, as will be the first-order perturbation theory shifts of the string energy eigenvalues. We defer the detailed discussion of this perturbation theory until we have the fermionic part of the problem in hand. Note that this discussion implies that if we wanted to determine the perturbed energies to higher orders in $1/R^2$, we would have the very unpleasant problem of dealing with the square root form of the Hamiltonian (2.12).

We have to this point been discussing a perturbative approach to finding the effect of the true geometry of the $AdS_5 \times S^5$ background on the string spectrum. Before proceeding with this program, however, it is instructive to study a different limit in which the kinematics are unrestricted (no large-J limit is taken) but only modes of the string that are independent of the worldsheet coordinate (the zero-modes of the string) are kept in the Hamiltonian. This is the problem of quantizing the superparticle of the underlying supergravity in the $AdS_5 \times S^5$ background, a problem which has been solved many times (for references, see [8]). A remarkable fact, which seems not to have been explicitly observed before, is that the spectrum of the zero-mode Hamiltonian is *exactly* a sum of harmonic oscillators: the curvature corrections we propose to compute actually vanish on this special subspace. This fact is important to an understanding of the full problem, so we will make a brief digression to explain the solution to this toy problem.

The quantization of the superparticle in a supergravity background is equivalent to finding the eigensolutions of certain Laplacians, one for each spin that occurs in the superparticle massless multiplet. The point of interest to us can be made by analyzing the dynamics of the scalar particle and its associated scalar Laplacian, which only depends on the background metric. With apologies, we will adopt another version of the $AdS_5 \times S^5$ metric, chosen because the scalar Laplacian is very simple in these coordinates:

$$\begin{aligned} ds^2 = -dt^2(R^2+z^2) + d\phi^2(R^2-y^2) + \\ dz^j\left(\delta_{jk} - \frac{z^j z^k}{R^2+z^2}\right)dz^k + dy^{j'}\left(\delta_{j'k'} + \frac{y^{j'}y^{k'}}{R^2-y^2}\right)dy^{k'} \ . \end{aligned} \tag{2.17}$$

As before, the coordinates z^k and $y^{k'}$ parameterize the two $SO(4)$ subspaces, and the indices j,k and j',k' run over $j,k=1,\ldots,4$, and $j',k'=5,\ldots,8$. This is a natural metric for analyzing fluctuations of a particle (or string) around the lightlike trajectory $\phi = t$ and $\vec{z} = \vec{y} = 0$. Because the metric components depend neither on t nor on ϕ, and because the problem is clearly separable in $\vec{z}$ and $\vec{y}$, it makes sense to look for solutions of the form $\Phi = e^{-i\omega t}e^{iJ\phi}F(\vec{z})G(\vec{y})$. The scalar Laplacian for ϕ in the above metric then reduces to

$$\begin{aligned} \Big[-\frac{\omega^2}{R^2+\vec{z}^2} + \frac{J^2}{R^2-\vec{y}^2} - \frac{\partial}{\partial x^j}\left(\delta^{jk} + \frac{z^j z^k}{R^2}\right)\frac{\partial}{\partial z^k} - \\ \frac{\partial}{\partial y^{j'}}\left(\delta^{j'k'} - \frac{y^{j'}y^{k'}}{R^2}\right)\frac{\partial}{\partial y^{k'}}\Big]F(z)G(y) = 0 \ . \end{aligned} \tag{2.18}$$

The radius R disappears from the equation upon rescaling the transverse coordinates by $z \to z/R$ and $y \to y/R$, so we can set $R=1$ in what follows and use dimensional analysis to restore R if it is needed. The scalar Laplacian is essentially the light-cone Hamiltonian constraint (2.10) for string coordinates $z^k, y^{k'}$ and string momenta $p_z^k = -i\frac{\partial}{\partial z^k}$ and $p_y^{k'} = -i\frac{\partial}{\partial y^{k'}}$ (projected onto their zero modes). This implies that we can use the structure of the Laplacian to correctly order operators in the string Hamiltonian.

The periodicity $\phi \equiv \phi + 2\pi$ means that the angular momentum J is integrally quantized. The allowed values of ω then follow from the solution of the eigenvalue problem posed by (2.18). As the trial function Φ indicates, (2.18) breaks into separate problems for $\vec{z}$ and $\vec{y}$:

$$\begin{aligned} \mathcal{H}_{AdS_5}F(\vec{z}) &= \left[p_j^z(\delta^{jk} + z^j z^k)p_k^z + \omega^2\frac{z_k z^k}{1+(z_k z^k)^2}\right]F(\vec{z}) = A(\omega)F(\vec{z}) \\ \mathcal{H}_{S^5}G(\vec{y}) &= \left[p_{j'}^y(\delta^{j'k'} - y^{j'}y^{k'})p_{k'}^y + J^2\frac{y_{k'}y^{k'}}{1-(y_{k'}y^{k'})^2}\right]G(\vec{y}) = B(J)G(\vec{y}) \ , \end{aligned} \tag{2.19}$$

where $\omega^2 - J^2 = A + B$. The separation eigenvalues A, B depend on their respective parameters ω, J, and we determine the energy eigenvalues ω by finding the roots of the potentially complicated equation $\omega^2 - J^2 - A - B = 0$. The scalar Laplacian (2.18) is equivalent to the constraint equation (2.10) projected onto string zero modes, and we are once again seeing

that the constraint doesn't directly give the Hamiltonian but rather an equation (quadratic or worse) to be solved for the Hamiltonian.

The $\mathcal{H}_{S^5}$ equation is just a repackaging of the problem of finding the eigenvalues of the $SO(6)$ Casimir invariant (another name for the scalar Laplacian on S^5) and $\mathcal{H}_{AdS_5}$ poses the corresponding problem for $SO(4,2)$. The $SO(6)$ eigenvalues are obviously discrete, and the $SO(4,2)$ problem also turns out to be discrete when one imposes the condition of finiteness at $z^2 \to \infty$ on the eigenfunctions (this is a natural restriction in the context of the AdS/CFT correspondence; for a detailed discussion see [8]). Thus we expect ω to have a purely discrete spectrum, with eigenvalues labeled by a set of integers. The simplest way to solve for the spectrum is to expand $F(\vec{z})$ and $G(\vec{y})$ in $SO(4)$ harmonics (since this symmetry is explicit), recognize that the radial equation is, in both cases, an example of Riemann's differential equation and then use known properties of the hypergeometric function to find the eigenvalues and eigenfunctions of (2.19). Since it takes three integers to specify an $SO(4)$ harmonic and one to specify a radial quantum number, we expect each of the two separated equations to have a spectrum labeled by four integers. The exact results for the separation eigenvalues turn out to be remarkably simple:

$$
\begin{aligned}
A &= 2\omega \sum_1^4 \left(n_i + \frac{1}{2}\right) - \left[\sum_1^4 \left(n_i + \frac{1}{2}\right)\right]^2 + 4 \qquad n_i = 0,1,2,\ldots \\
B &= 2J \sum_1^4 \left(m_i + \frac{1}{2}\right) + \left[\sum_1^4 \left(m_i + \frac{1}{2}\right)\right]^2 + 4 \qquad m_i = 0,1,2,\ldots
\end{aligned} \tag{2.20}
$$

Different eigenfunctions correspond to different choices of the collection of eight integers $\{n_i, m_i\}$, and the fact that the energies depend only on Σn_i and Σm_i correctly accounts for the degeneracy of eigenvalues. The special form of A and B means that the equation for the energy eigenvalue, $\omega^2 - J^2 - A - B = 0$, can be factored as

$$
\begin{aligned}
&\left[\omega - J - \sum_1^4 \left(n_i + \frac{1}{2}\right) - \sum_1^4 \left(m_i + \frac{1}{2}\right)\right] \\
&\qquad \times \left[\omega + J - \sum_1^4 \left(n_i + \frac{1}{2}\right) + \sum_1^4 \left(m_i + \frac{1}{2}\right)\right] = 0 \ .
\end{aligned} \tag{2.21}
$$

For obvious reasons, we retain the root that assigns only positive values to ω, the energy conjugate to the global time t:

$$
\omega - J = \sum_1^4 \left(n_i + \frac{1}{2}\right) + \sum_1^4 \left(m_i + \frac{1}{2}\right) \ . \tag{2.22}
$$

From the string point of view, ω catalogs the eigenvalues of the string worldsheet Hamiltonian restricted to the zero-mode subspace. Quite remarkably, it is an exact 'sum of harmonic

oscillators', independent of whether J (and ω) are large or not. This is simply to say that the eigenvalues of the string Hamiltonian restricted to the zero-mode sector receive no curvature corrections and could have been calculated from the pp-wave string Hamiltonian (2.15). We have only shown this for the massless bosons of the theory, but we expect the same thing to be true for all the massless fields of type IIB supergravity. The implication for a perturbative account of the string spectrum is that states created using only zero-mode oscillators (of any type) will receive no curvature corrections. This feature will turn out to be a useful consistency check on our quantization procedure. It is of course not true for a general classical background and is yet another manifestation of the special nature of the $AdS_5 \times S^5$ geometry.

3 GS superstring action on $AdS_5 \times S^5$

The $AdS_5 \times S^5$ target space can be realized as the coset superspace

$$G/H = \frac{SU(2,2|4)}{SO(4,1) \times SO(5)} . \tag{3.1}$$

The bosonic reduction of this coset is precisely $SO(4,2) \times SO(6)/SO(4,1) \times SO(5) \equiv AdS_5 \times S^5$. To quantize the theory, we will expand the action about a classical trajectory which happens to be invariant under the stabilizer group H. There is a general strategy for constructing a non-linear sigma model on a super-coset space in terms of the Cartan one-forms and superconnections of the super-coset manifold. In such a construction, the symmetries of the stabilizer subgroup remain manifest in the action while the remaining symmetries are nonlinearly realized (see, e.g., [9, 10, 7, 11, 12, 13]). Metsaev and Tseytlin [10] carried out this construction for the $AdS_5 \times S^5$ geometry, producing a κ-symmetric, type IIB superstring action possessing the full $PSU(2,2|4)$ supersymmetry of $AdS_5 \times S^5$. Their action is conceptually simple, comprising a kinetic term and a Wess–Zumino term built out of Cartan (super)one-forms on the super-coset manifold in the following way (this form was first presented in [14]):

$$\mathcal{S} \;=\; -\frac{1}{2}\int_{\partial M_3} d^2\sigma\; h^{ab} L_a^\mu L_b^\mu + i\int_{M_3} s^{IJ} L^\mu \wedge \bar{L}^I \Gamma^\mu \wedge L^J . \tag{3.2}$$

Repeated upper indices are summed over a Minkowskian inner product. The indices a, b are used to indicate the worldsheet coordinates (τ, σ), and we use the values $a, b = 0$ to indicate the worldsheet time direction τ, and $a, b = 1$ to specify the σ direction. The matrix s^{IJ} is defined by $s^{IJ} \equiv \mathrm{diag}(1, -1)$, where $I, J = 1, 2$. The Wess-Zumino term appears as an integral over a 3-manifold M_3, while the kinetic term is integrated over the 2-dimensional boundary ∂M_3. The left-invariant Cartan forms are defined in terms of the coset space representative G by

$$G^{-1}dG = L^\mu P^\mu + L^\alpha \bar{Q}_\alpha + \bar{L}^\alpha Q_\alpha + \frac{1}{2} L^{\mu\nu} J^{\mu\nu}$$
$$L^N = dX^M L_M^N \qquad L_a^N = L_M^N \partial_a X^M \qquad X^M = (x^\mu, \theta^\alpha, \bar{\theta}^\alpha) , \tag{3.3}$$

The explicit expansion of this action in terms of independent fermionic degrees of freedom is rather intricate. One starts with two 32-component Majorana-Weyl spinors in 10 dimensions: θ^I, where $I = 1, 2$ labels the two spinors. In a suitably-chosen representation for the 32×32 ten-dimensional gamma matrices Γ^μ, the Weyl projection reduces to picking out the upper 16 components of θ and the surviving spinors can combined into one complex 16-component spinor ψ:

$$\theta^I = \begin{pmatrix} \theta^\alpha \\ 0 \end{pmatrix}^I \qquad\qquad \psi^\alpha = \sqrt{2}\left[(\theta^\alpha)^1 + i(\theta^\alpha)^2\right] \ . \tag{3.4}$$

The following representation for Γ^μ (which has the desired property that $\Gamma_{11} = (\mathbf{1}_8, -\mathbf{1}_8)$) allows us to express their action on ψ in terms of real 16×16 γ-matrices:

$$\begin{aligned} \Gamma^\mu &= \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix} \qquad\qquad && \gamma^\mu\bar{\gamma}^\nu + \gamma^\nu\bar{\gamma}^\mu = 2\eta^{\mu\nu} \\ \gamma^\mu &= (1, \gamma^A, \gamma^9) && \bar{\gamma}^\mu = (-1, \gamma^A, \gamma^9) \ . \end{aligned} \tag{3.5}$$

The indices $\mu, \nu, \rho = 0, \ldots, 9$ denote $SO(9,1)$ vectors, and we will denote the corresponding spinor indices by $\alpha, \beta, \gamma, \delta = 1, \ldots, 16$ (we also use the convention that upper-case indices $A, B, C, D = 1, \ldots, 8$ indicate vectors of $SO(8)$, while $i, j, k = 1, \ldots, 4$ ($i', j', k' = 5, \ldots, 8$) indicate vectors from the $SO(3,1) \cong SO(4)$ ($SO(4)$) subspaces associated with AdS_5 and S^5 respectively). The matrix γ^9 is formed by taking the product of the eight γ^A. A representation of γ^A matrices which will be convenient for explicit calculation is given in Appendix A. We also note that in the course of quantization we will impose the fermionic lightcone gauge fixing condition $\bar{\gamma}^9\psi = \psi$. This restricts the worldsheet fermions to lie in the 8_s representation of $SO(8)$ (and projects out the 8_c spinor), thus reducing the number of independent components of the worldsheet spinor from 16 to 8. The symmetric matrix

$$\Pi \equiv \gamma^1\bar{\gamma}^2\gamma^3\bar{\gamma}^4 \tag{3.6}$$

appears in a number of places in the expansion of the action, so we give it an explicit definition. Since $\Pi^2 = 1$, it has eigenvalues ± 1 which turn out to provide a useful subclassification of the 8 components of the 8_s worldsheet spinor into two groups of 4. The quantity $\tilde{\Pi} = \Pi\gamma_9$ also appears, but does not require a separate definition because $\Pi\psi = \tilde{\Pi}\psi$ for spinors satisfying the lightcone gauge restriction to the 8_s representation.

Kallosh, Rahmfeld and Rajaraman presented in [9] a general solution to the supergravity constraints (Maurer-Cartan equations) for coset spaces exhibiting a superconformal isometry algebra of the form

$$\begin{aligned} [B_\mu, B_\nu] &= f^\rho_{\mu\nu} B_\rho \\ [F_\alpha, B_\nu] &= f^\beta_{\alpha\nu} F_\beta \\ \{F_\alpha, F_\beta\} &= f^\mu_{\alpha\beta} B_\mu \ , \end{aligned} \tag{3.7}$$

with B_μ and F_α representing bosonic and fermionic generators, respectively. In terms of these generators, the Cartan forms L^μ and superconnections L^α are determined completely by the structure constants $f^J_{\alpha\mu}$ and $f^\mu_{\alpha\beta}$:

$$
\begin{aligned}
L^\alpha_{at} &= \left(\frac{\sinh t\mathcal{M}}{\mathcal{M}}\right)^\alpha_\beta (\mathcal{D}_a\theta)^\beta && (3.8)\\
L^\mu_{at} &= e^\mu{}_\nu \partial_a x^\nu + 2\theta^\alpha f^\mu_{\alpha\beta}\left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2}\right)^\beta_\gamma (\mathcal{D}_a\theta)^\gamma && (3.9)\\
(\mathcal{M}^2)^\alpha_\beta &= -\theta^\gamma f^\alpha_{\gamma\mu}\theta^\delta f^\mu_{\delta\beta}\ . && (3.10)
\end{aligned}
$$

The dimensionless parameter t is used here to define "shifted" Cartan forms and superconnections where, for example, $L^\mu_a = L^\mu_{at}|_{t=1}$. In the case of $AdS_5 \times S^5$, the Lagrangian takes the form

$$
\begin{aligned}
\mathcal{L}_{\text{Kin}} &= -\frac{1}{2} h^{ab} L^\mu_a L^\mu_b && (3.11)\\
\mathcal{L}_{\text{WZ}} &= -2i\epsilon^{ab}\int_0^1 dt\, L^\mu_{at} s^{IJ}\bar{\theta}^I \Gamma^\mu L^J_{bt}\ . && (3.12)
\end{aligned}
$$

In the context of eqns. (3.8,3.9), it will be useful to choose a manifestation of the spacetime metric that yields a compact form of the spin connection. The form appearing in eqn. (2.3) is well suited to this requirement; the AdS_5 and S^5 subspaces are represented in (2.3) nearly symmetrically, and the spin connection is relatively simple:

$$
\begin{aligned}
\omega^{t\,z_k}{}_t &= \frac{z_k}{1-\frac{1}{4}z^2} & \omega^{z_j\,z_k}{}_{z_j} &= \frac{\frac{1}{2}z_k}{1-\frac{1}{4}z^2}\\
\omega^{\phi\,y_{k'}}{}_\phi &= -\frac{y_{k'}}{1+\frac{1}{4}y^2} & \omega^{y_{j'}\,y_{k'}}{}_{y_{j'}} &= -\frac{\frac{1}{2}y_{k'}}{1+\frac{1}{4}y^2}\ . && (3.13)
\end{aligned}
$$

Upon moving to the light-cone coordinate system in (2.4), the x^+ direction remains null ($G_{--}=0$) to $\mathcal{O}(1/R^4)$ in this expansion.

By introducing dimensionless contraction parameters Λ and Ω [15], one may express the

$AdS_5 \times S^5$ isometry algebra keeping light-cone directions explicit:

$$
\begin{array}{rl}
\left[P^+, P^k\right] = \Lambda^2\Omega^2 J^{+k} & \left[P^+, P^{k'}\right] = -\Lambda^2\Omega^2 J^{+k'} \\
\left[P^+, J^{+k}\right] = -\Lambda^2 P^k & \left[P^+, J^{+k'}\right] = \Lambda^2 P^{k'} \\
\left[P^-, P^A\right] = \Omega^2 J^{+A} & \left[P^-, J^{+A}\right] = P^A \\
\left[P^j, P^k\right] = \Lambda^2\Omega^2 J^{jk} & \left[P^{j'}, P^{k'}\right] = -\Lambda^2\Omega^2 J^{j'k'} \\
\left[J^{+j}, J^{+k}\right] = \Lambda^2 J^{jk} & \left[J^{+j'}, J^{+k'}\right] = -\Lambda^2 J^{j'k'} \\
\left[P^j, J^{+k}\right] = -\delta^{jk}(P^+ - \Lambda^2 P^-) & \left[P^r, J^{+s}\right] = -\delta^{rs}(P^+ + \Lambda^2 P^-) \\
\left[P^i, J^{jk}\right] = \delta^{ij} P^k - \delta^{ik} P^j & \left[P^{i'}, J^{j'k'}\right] = \delta^{i'j'} P^{k'} - \delta^{i'j'} P^{k'} \\
\left[J^{+i}, J^{jk}\right] = \delta^{ij} J^{+k} - \delta^{ik} J^{+j} & \left[J^{+i'}, J^{j'k'}\right] = \delta^{i'j'} J^{+k'} - \delta^{i'j'} J^{+k'} \\
\left[J^{ij}, J^{kl}\right] = \delta^{jk} J^{il} + 3 \text{ terms} & \left[J^{i'j'}, J^{k'l'}\right] = \delta^{j'k'} J^{i'l'} + 3 \text{ terms} .
\end{array} \tag{3.14}
$$

The bosonic sector of the algebra relevant to (3.7) takes the form

$$
\begin{aligned}
\left[J^{ij}, Q_\alpha\right] &= \frac{1}{2} Q_\beta (\gamma^{ij})^\beta{}_\alpha \\
\left[J^{i'j'}, Q_\alpha\right] &= \frac{1}{2} Q_\beta (\gamma^{i'j'})^\beta{}_\alpha \\
\left[J^{+i}, Q_\alpha\right] &= \frac{1}{2} Q_\beta (\gamma^{+i} - \Lambda^2 \gamma^{-i})^\beta{}_\alpha \\
\left[J^{+i'}, Q_\alpha\right] &= \frac{1}{2} Q_\beta (\gamma^{+i'} + \Lambda^2 \gamma^{-i'})^\beta{}_\alpha \\
\left[P^\mu, Q_\alpha\right] &= \frac{i\Omega}{2} Q_\beta (\Pi\gamma^+\bar\gamma^\mu)^\beta{}_\alpha - \frac{i\Lambda^2\Omega}{2} Q_\beta (\Pi\gamma^-\bar\gamma^\mu)^\beta{}_\alpha .
\end{aligned} \tag{3.15}
$$

The fermi-fermi anticommutation relations are

$$
\begin{aligned}
\{Q_\alpha, \bar{Q}_\beta\} = {} & -2i\gamma^\mu_{\alpha\beta} P^\mu - 2\Omega(\bar\gamma^k\Pi)_{\alpha\beta} J^{+k} - 2\Omega(\bar\gamma^{k'}\Pi)_{\alpha\beta} J^{+k'} \\
& +\Omega(\bar\gamma^+\gamma^{jk}\Pi)_{\alpha\beta} J^{jk} + \Omega(\bar\gamma^+\gamma^{j'k'}\Pi)_{\alpha\beta} J^{j'k'} \\
& -\Lambda^2\Omega(\bar\gamma^-\gamma^{jk}\Pi)_{\alpha\beta} J^{jk} + \Lambda^2\Omega(\bar\gamma^-\gamma^{j'k'}\Pi)_{\alpha\beta} J^{j'k'} .
\end{aligned} \tag{3.16}
$$

This form of the superalgebra has the virtue that one can easily identify the flat space ($\Omega \to 0$) and plane-wave ($\Lambda \to 0$) limits. The Maurer-Cartan equations in this coordinate system take the form

$$
\begin{aligned}
dL^\mu &= -L^{\mu\nu} L^\nu - 2i\bar{L}\bar\gamma^\mu L \\
dL^\alpha &= -\frac{1}{4} L^{\mu\nu} (\gamma^{\mu\nu})^\alpha{}_\beta L^\beta + \frac{i\Omega}{2} L^\mu (\Pi\gamma^+\bar\gamma^\mu)^\alpha{}_\beta L^\beta - \frac{i\Lambda^2\Omega}{2} L^\mu (\Pi\gamma^-\bar\gamma^\mu)^\alpha{}_\beta L^\beta \\
d\bar{L}^\alpha &= -\frac{1}{4} L^{\mu\nu} (\gamma^{\mu\nu})^\alpha{}_\beta \bar{L}^\beta - \frac{i\Omega}{2} L^\mu (\Pi\gamma^+\bar\gamma^\mu)^\alpha{}_\beta \bar{L}^\beta + \frac{i\Lambda^2\Omega}{2} L^\mu (\Pi\gamma^-\bar\gamma^\mu)^\alpha{}_\beta \bar{L}^\beta ,
\end{aligned} \tag{3.17}
$$

where wedge products (3.2) are understood to be replaced by the following rules:

$$L^\mu L^\nu = -L^\nu L^\mu \qquad L^\mu L^\alpha = -L^\alpha L^\mu \qquad L^\alpha L^\beta = L^\beta L^\alpha \ . \tag{3.18}$$

Upon choosing a parameterization of the coset representative G

$$G(x,\theta) = f(x)g(\theta) \qquad g(\theta) = \exp(\theta^\alpha \bar{Q}_\alpha + \bar{\theta}^\alpha Q_\alpha) \ , \tag{3.19}$$

one derives a set of coupled differential equations for the shifted Cartan forms and super-connections:

$$\begin{aligned}
\partial_t L_t &= d\theta + \frac{1}{4} L_t^{\mu\nu}\gamma^{\mu\nu}\theta - \frac{i\Omega}{2} L_t^\mu \Pi\gamma^+\bar{\gamma}^\mu\theta + \frac{i\Lambda^2\Omega}{2} L_t^\mu \Pi\gamma^-\bar{\gamma}^\mu\theta \\
\partial_t L_t^\mu &= -2i\theta\bar{\gamma}^\mu \bar{L}_t - 2i\bar{\theta}\bar{\gamma}^\mu L_t \\
\partial_t L_t^{-i} &= 2\Omega(\theta\bar{\gamma}^i\Pi\bar{L}_t) - 2\Omega(\bar{\theta}\bar{\gamma}^i\Pi L_t) \\
\partial_t L_t^{-r} &= 2\Omega(\theta\bar{\gamma}^r\Pi\bar{L}_t) - 2\Omega(\bar{\theta}\bar{\gamma}^r\Pi L_t) \\
\partial_t L_t^{ij} &= -2\Omega(\theta\bar{\gamma}^+\gamma^{ij}\Pi\bar{L}_t) + 2\Omega(\bar{\theta}\bar{\gamma}^+\gamma^{ij}\Pi L_t) + 2\Lambda^2\Omega(\theta\bar{\gamma}^-\gamma^{ij}\Pi\bar{L}_t) - 2\Lambda^2\Omega(\bar{\theta}\bar{\gamma}^-\gamma^{ij}\Pi L_t) \\
\partial_t L_t^{i'j'} &= -2\Omega(\theta\bar{\gamma}^+\gamma^{i'j'}\Pi\bar{L}_t) + 2\Omega(\bar{\theta}\bar{\gamma}^+\gamma^{i'j'}\Pi L_t) - 2\Lambda^2\Omega(\theta\bar{\gamma}^-\gamma^{i'j'}\Pi\bar{L}_t) \\
&\quad + 2\Lambda^2\Omega(\bar{\theta}\bar{\gamma}^-\gamma^{i'j'}\Pi L_t) \ .
\end{aligned} \tag{3.20}$$

These coupled equations are subject to the following boundary conditions:

$$\begin{aligned}
L_\pm(t=0) = 0 \quad L^\mu_{t=0} = e^\mu \quad L^\pm_{t=0} = e^\pm \\
L^{\mu\nu}_{t=0} = \omega^{\mu\nu} \qquad L^{-\mu}_{t=0} = \omega^{-\mu} \ .
\end{aligned} \tag{3.21}$$

The generators $J^{-\mu}$ and $J^{kk'}$ are not present in the superalgebra, so the conditions

$$L^{+\mu} = 0 \qquad L^{kk'} = 0 \tag{3.22}$$

are imposed as constraints.

To employ the general solution to the Maurer-Cartan equations (3.8,3.9), the relevant sectors of the superalgebra may be rewritten in the more convenient 32-dimensional notation (setting $\Lambda = 1$ and $\Omega = 1$):

$$\begin{aligned}
[Q_I, P^\mu] &= \frac{i}{2}\epsilon^{IJ} Q_J \Gamma_* \Gamma^\mu \\
[Q_I, J^{\mu\nu}] &= -\frac{1}{2} Q_I \Gamma^{\mu\nu} \\
\{(Q_I)^\mu, (Q_J)_\mu\} &= -2i\delta_{IJ}\Gamma^0\Gamma^\rho P_\rho + \epsilon^{IJ}\left(-\Gamma^0\Gamma^{jk}\Gamma_* J_{jk} + \Gamma^0\Gamma^{j'k'}\Gamma'_* J_{j'k'}\right) \ ,
\end{aligned} \tag{3.23}$$

where

$$\Gamma_* \equiv i\Gamma_{01234} \qquad \Gamma'_* \equiv i\Gamma_{56789} \ . \tag{3.24}$$

The Cartan forms and superconnections then take the following form:

$$L^J_{bt} = \frac{\sinh t\mathcal{M}}{\mathcal{M}}\mathcal{D}_b\theta^J \qquad\qquad L^\mu_{at} = e^\mu{}_\rho\partial_a x^\rho - 4i\bar\theta^I\Gamma^\mu\left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2}\right)\mathcal{D}_a\theta^I \ , \quad (3.25)$$

where the covariant derivative is given by

$$(\mathcal{D}_a\theta)^I \ = \ \left(\partial_a\theta + \frac{1}{4}\left(\omega^{\mu\nu}{}_\rho\,\partial_a x^\rho\right)\Gamma^{\mu\nu}\theta\right)^I - \frac{i}{2}\epsilon^{IJ}e^\mu{}_\rho\,\partial_a x^\rho\Gamma_*\Gamma^\mu\theta^J \ . \qquad (3.26)$$

The object $\mathcal{M}$ is a 2×2 matrix which, for convenience, is defined in terms of its square:

$$(\mathcal{M}^2)^{IL} \ = \ -\epsilon^{IJ}(\Gamma_*\Gamma^\mu\theta^J\bar\theta^L\Gamma^\mu) + \frac{1}{2}\epsilon^{KL}(-\Gamma^{jk}\theta^I\bar\theta^K\Gamma^{jk}\Gamma_* + \Gamma^{j'k'}\theta^I\bar\theta^K\Gamma^{j'k'}\Gamma'_*) \ . \ (3.27)$$

At this point, the GS action on $AdS_5\times S^5$ (3.11,3.12) may be expanded to arbitrary order in fermionic and bosonic fields. In the present calculation, the parameters Ω and Λ remain set to unity, and the action is expanded in inverse powers of the target-space radius R, introduced in the rescaled light-cone coordinates in eqn. (2.4). The fact that supersymmetry must be protected at each order in the expansion determines a rescaling prescription for the fermions. Accordingly, the eight transverse bosonic directions x^A and the corresponding fermionic fields ψ^α receive a rescaling coefficient proportional to R^{-1}. The first curvature correction away from the plane-wave limit therefore occurs at quartic order in both bosonic and fermionic fluctuations. The particular light-cone coordinate system chosen in (2.4), however, gives rise to several complications. The $x^\pm$ coordinates given by

$$t = x^+ - \frac{x^-}{2R^2} \qquad\qquad \phi = x^+ + \frac{x^-}{2R^2} \qquad (3.28)$$

have conjugate momenta (in the language of BMN)

$$-p_+ \ = \ i\partial_{x^+} = i(\partial_t + \partial_\phi) = \Delta - J \qquad (3.29)$$

$$-p_- \ = \ i\partial_{x^-} = \frac{i}{2R^2}(\partial_\phi - \partial_t) = -\frac{1}{2R^2}(\Delta + J) \ , \qquad (3.30)$$

with $\Delta = E = i\partial_t$ and $J = -i\partial_\phi$. The light-cone Hamiltonian is $\mathcal{H} = -p_+$, so with $\Delta = J - p_+$ one may schematically write

$$\begin{aligned} p_- \ &= \ \frac{1}{2R^2}(2J - p_+) \\ &= \ \frac{J}{R^2} + \frac{\mathcal{H}}{2R^2} \\ &= \ \frac{J}{R^2}\left(1 + \frac{1}{2J}\sum N\omega\right) \ . \end{aligned} \qquad (3.31)$$

This result appears to be incorrect in the context of the light-cone gauge condition $\partial_\tau t = p_-$. To compensate for this, one must set the constant worldsheet density p_- equal to

something different from 1 (and non-constant) if the parameter length of the worldsheet is to be proportional to J. This operation introduces an additional $\mathcal{O}(1/R^2)$ shift in the energy of the string oscillators. This is acceptable because, in practice, we wish to consider only degenerate subsets of energy states for comparison between the gauge theory and string theory results. Because of the compensation between corrections to J and the Hamiltonian contribution from p_-, the eigenvalues of J will remain constant within these degenerate subsets. Therefore, while it may seem incorrect to introduce operator-valued corrections to p_-, one could proceed pragmatically with the intent of restricting oneself to these degenerate subsets.[1]

A different choice of light-cone coordinates allows us to avoid this problem completely. By choosing

$$\begin{aligned} t &= x^+ \\ \phi &= x^+ + \frac{x^-}{R^2} \, , \end{aligned} \tag{3.32}$$

we have

$$-p_+ = \Delta - J \tag{3.33}$$

$$-p_- = i\partial_{x^-} = \frac{i}{R^2}\partial_\phi = -\frac{J}{R^2} \, , \tag{3.34}$$

such that p_- appears as a legitimate expansion parameter in the theory. In this coordinate system, the curvature expansion of the metric becomes

$$\begin{aligned} ds^2 \approx\ & 2dx^+dx^- - (x^A)^2(dx^+)^2 + (dx^A)^2 \\ & + \frac{1}{R^2}\left[-2y^2dx^+dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2}z^2dz^2 - \frac{1}{2}y^2dy^2\right] \\ & + \mathcal{O}\left(R^{-4}\right) . \end{aligned} \tag{3.35}$$

The operator-valued terms in p_- that appear under the first coordinate choice (3.28) are no longer present. However, it will be shown that this new coordinate system induces correction terms to the spacetime curvature of the worldsheet metric. Furthermore, the appearance of a nonvanishing G_{--} component, and the loss of many convenient symmetries between terms associated with the x^+ and x^- directions bring some additional complications into the analysis. The advantage is that the results will be unambiguous in the end (and free from normal-ordering ambiguities).

4 Curvature corrections to the Penrose limit

In this section we expand the GS superstring action on $AdS_5 \times S^5$ in powers of $1/R^2$. We begin by constructing various quantities including combinations of Cartan 1-forms relevant

[1] When such a program is carried out, however, the resulting theory is subject to normal-ordering ambiguities; we instead use a coordinate system that is free of these complications.

to the worldsheet Lagrangian. Spacetime curvature corrections to the worldsheet metric will be calculated by analyzing the x^- equation of motion and the covariant gauge constraints order-by-order.

We introduce the notation

$$\Delta^\mu_n \equiv \bar\theta^I \Gamma^\mu \mathcal{D}^n_0 \theta^I \tag{4.1}$$
$$\Delta'^\mu_n \equiv \bar\theta^I \Gamma^\mu \mathcal{D}^n_1 \theta^I \ , \tag{4.2}$$

where the covariant derivative is expanded in powers of $(1/R)$:

$$\mathcal{D}_a = \mathcal{D}^0_a + \frac{1}{R}\mathcal{D}^1_a + \frac{1}{R^2}\mathcal{D}^2_a + \mathcal{O}(R^{-3}) \ . \tag{4.3}$$

Terms in the Wess-Zumino Lagrangian are encoded using a similar notation:

$$\Box^\mu_n \equiv s^{IJ} \bar\theta^I \Gamma^\mu \mathcal{D}^n_0 \theta^J \tag{4.4}$$
$$\Box'^\mu_n \equiv s^{IJ} \bar\theta^I \Gamma^\mu \mathcal{D}^n_1 \theta^J \ . \tag{4.5}$$

The subscript notation $(\Delta^\mu_n)_{\theta^4}$ will be used to indicate the quartic fermionic term involving $\mathcal{M}^2$:

$$(\Delta^\mu_n)_{\theta^4} \equiv \frac{1}{12}\bar\theta^I(\mathcal{M}^2)\mathcal{D}^n_0\theta^I \ . \tag{4.6}$$

For the present, it will be convenient to remove an overall factor of R^2 from the definition of the vielbeins $e^\mu{}_\nu$. In practice, this choice makes it easier to recognize terms that contribute to the Hamiltonian at the order of interest, and, in the end, allows us to avoid imposing an additional rescaling operation on the fermions. We proceed by keeping terms to $\mathcal{O}(1/R^4)$, with the understanding that an extra factor of R^2 must be removed in the final analysis. The covariant derivative

$$\mathcal{D}_a\theta^I = \partial_a\theta^I + \frac{1}{4}\partial_a x^\mu \omega^{\nu\rho}_\mu \Gamma_{\nu\rho}\theta^I - \frac{i}{2}\epsilon^{IJ}\Gamma_*\Gamma_\mu e^\mu{}_\nu \partial_a x^\nu \theta^J \tag{4.7}$$

may then be expanded to $\mathcal{O}(1/R^2)$ (we will not need $\mathcal{O}(1/R^3)$ terms, because the covariant derivative always appears left-multiplied by a spacetime spinor $\bar\theta$):

$$\begin{aligned}\mathcal{D}_0\theta^I = {}& \left[\partial_0\theta^I - p_-\epsilon^{IJ}\Pi\theta^J\right] + \frac{1}{R}\left[\frac{p_-}{4}\left(z_j\Gamma^{-j} - y_{j'}\Gamma^{-j'}\right)\theta^I + \frac{1}{4}\epsilon^{IJ}\Gamma^-\Pi(\dot x^A\Gamma^A)\theta^J\right] \\ & + \frac{1}{R^2}\left[\frac{1}{4}(\dot z_j z_k\Gamma^{jk} - \dot y_{j'}y_{k'}\Gamma^{j'k'})\theta^I + \frac{p_-}{4}\epsilon^{IJ}\Pi(y^2 - z^2)\theta^J - \frac{1}{2}\epsilon^{IJ}(\dot x^-)\Pi\theta^J\right] \\ & + \mathcal{O}(R^{-3})\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathcal{D}_1\theta^I = {}& \partial_1\theta^I + \frac{1}{4R}\epsilon^{IJ}\Gamma^-\Pi(x'^A\Gamma^A)\theta^J \\ & + \frac{1}{R^2}\left[\frac{1}{4}(z'_j z_k\Gamma^{jk} - y'_{j'}y_{k'}\Gamma^{j'k'})\theta^I - \frac{1}{2}\epsilon^{IJ}(x'^-)\Pi\theta^J\right] + \mathcal{O}(R^{-3}) \ .\end{aligned} \tag{4.9}$$

Note that we have not rescaled the spinor field θ in the above expansion. This allows us to isolate the bosonic scaling contribution from the covariant derivative when combining various terms in the Lagrangian. Subsequently, the fermionic rescaling is performed based on the number of spinors appearing in each term (two spinors for each Δ^μ or $\Box^\mu$, and four for each $(\Delta^\mu)_{\theta^4}$). The worldsheet derivative notation is given by $\partial_\tau x = \partial_0 x = \dot{x}$ and $\partial_\sigma x = \partial_1 x = x'$.

The various sectors of the worldsheet Lagrangian are assembled keeping x^- and its derivatives explicit; these will be removed by imposing the covariant gauge constraints. From the supervielbein and superconnection

$$
\begin{aligned}
L^\mu_{at} &= e^\mu{}_\nu \partial_a x^\nu - 4i\bar{\theta}^I \Gamma^\mu \left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2} \right) \mathcal{D}_a \theta^I \\
&\approx e^\mu{}_\nu \partial_a x^\nu - i\bar{\theta}^I \Gamma^\mu \left(t^2 + \frac{t^4 \mathcal{M}^2}{12} \right) \mathcal{D}_a \theta^I \qquad (4.10) \\
L^I_{at} &= \frac{\sinh t\mathcal{M}}{\mathcal{M}} \mathcal{D}_a \theta^I \approx \left(t + \frac{t^3}{6}\mathcal{M}^2 \right) \mathcal{D}_a \theta^I \ , \qquad (4.11)
\end{aligned}
$$

we form the following objects:

$$
\begin{aligned}
L^\mu_0 L^\mu_0 &= \frac{1}{R^2} \left\{ 2p_- \dot{x}^- - p_-^2 (x^A)^2 + (\dot{x}^A)^2 - 2ip_- \Delta_0^- \right\} \\
&+ \frac{1}{R^4} \Bigg\{ (\dot{x}^-)^2 - 2p_- y^2 \dot{x}^- + \frac{1}{2}(\dot{z}^2 z^2 - \dot{y}^2 y^2) + \frac{p_-^2}{2}(y^4 - z^4) \\
&-2i \left[\frac{1}{2}\dot{x}^- \Delta_0^- + p_- \Delta_2^- + p_- (\Delta_0^-)_{\theta^4} - \frac{p_-}{4}(y^2 - z^2)\Delta_0^- + \dot{x}^A \Delta_1^A \right] \Bigg\} + \mathcal{O}(R^{-6}) \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
L^\mu_1 L^\mu_1 &= \frac{1}{R^2}(x'^A)^2 \\
&+ \frac{1}{R^4} \left\{ \frac{1}{2}(z'^2 z^2 - y'^2 y^2) + (x'^-)^2 - 2ix'^A \Delta'^A_1 - ix'^- \Delta'^-_0 \right\} + \mathcal{O}(R^{-6}) \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
L^\mu_0 L^\mu_1 &= \frac{1}{R^2} \left\{ p_- x'^- + \dot{x}^A x'^A - ip_- \Delta'^-_0 \right\} \\
&+ \frac{1}{R^4} \Bigg\{ x'^- \dot{x}^- - p_- y^2 x'^- + \frac{1}{2}(z^2 \dot{z}_k z'_k - y^2 \dot{y}_{k'} y'_{k'}) - ip_- \Delta'^-_2 - ip_- (\Delta'^-_0)_{\theta^4} \\
&- i\frac{p_-}{4}(z^2 - y^2)\Delta'^-_0 - \frac{i}{2}\dot{x}^- \Delta'^-_0 - i\dot{x}^A \Delta'^A_1 - ix'^A \Delta^A_1 - \frac{i}{2} x'^- \Delta^-_0 \Bigg\} + \mathcal{O}(R^{-6}) \ . \qquad (4.14)
\end{aligned}
$$

It will be advantageous to enforce the light-cone gauge condition $x^+ = \tau$ at all orders in the theory.[2] When fermions are included, this choice allows us to keep the κ-symmetry

[2] This differs from the approach presented in [6].

condition $\Gamma^+\theta = 0$ exact. In the pp-wave limit, keeping the worldsheet metric flat in this light-cone gauge is consistent with the equations of motion. Beyond leading order, however, we are forced to consider curvature corrections to the worldsheet metric that appear in both the conformal gauge constraints and the worldsheet Hamiltonian. In the purely bosonic case described in section 2 above, these corrections are kept implicit by defining gauge constraints in terms of canonical momenta. In the supersymmetric theory, we must explicitly calculate these corrections. The strategy is to expand the x^- equations of motion in rescaled coordinates (3.32) and solve for the components of the worldsheet metric order-by-order. By varying x^- in the full Lagrangian we obtain

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\dot{x}^-} &= \frac{1}{2}h^{00}\left\{\frac{2p_-}{R^2} + \frac{1}{R^4}\left[2\dot{x}^- - 2p_-y^2 - i\bar{\theta}^I\Gamma^-\partial_0\theta^I + 2ip_-\bar{\theta}^I\Gamma^-\epsilon^{IJ}\Pi\theta^J\right]\right\} \\ &\quad + \frac{i}{2R^4}s^{IJ}\bar{\theta}^I\Gamma^-\partial_1\theta^J + \mathcal{O}(R^{-6}). \end{aligned} \tag{4.15}$$

The worldsheet metric is taken to be flat at leading order, so there is no contribution from $L_0^\mu L_1^\mu$ here. To obtain corrections to h^{ab} entirely in terms of physical variables, however, we must eliminate all instances of x^- (or its derivatives) from the above variation. We can solve the conformal gauge constraints at leading order to remove $\dot{x}^-$ from (4.15). These constraints are obtained by varying the Lagrangian with respect to the worldsheet metric itself:

$$T_{ab} = L_a^\mu L_b^\mu - \frac{1}{2}h_{ab}h^{cd}L_c^\mu L_d^\mu \ , \tag{4.16}$$

yielding a symmetric traceless tensor with two independent components. To leading order in $1/R$, we find

$$\begin{aligned} T_{00} &= \frac{1}{2}(L_0^\mu L_0^\mu + L_1^\mu L_1^\mu) + \cdots = 0 \\ &= \frac{1}{2R^2}\Big(2p_-\dot{x}^- - p_-^2(x^A)^2 + (\dot{x}^A)^2 - 2ip_-\Delta_0^- + (x'^A)^2\Big) + \mathcal{O}(R^{-4}) \qquad (4.17) \\ T_{01} &= L_0^\mu L_1^\mu + \cdots = 0 \\ &= p_-x'^- + \dot{x}^A x'^A - ip_-\Delta_0'^- + \mathcal{O}(R^{-4}) \ . \qquad (4.18) \end{aligned}$$

Expanding $\dot{x}^-$ and x'^- in the same fashion,

$$\dot{x}^- = \sum_n \frac{a_n}{R^n} \qquad x'^- = \sum_n \frac{a'_n}{R^n} \ , \tag{4.19}$$

we use (4.17) and (4.18) to obtain

$$\begin{aligned} a_0 &= \frac{p_-}{2}(x^A)^2 - \frac{1}{2p_-}\left[(\dot{x}^A)^2 + (x'^A)^2\right] + i\bar{\theta}^I\Gamma^-\partial_0\theta^I - ip_-\epsilon^{IJ}\bar{\theta}^I\Gamma^-\Pi\theta^J \qquad (4.20) \\ a'_0 &= -\frac{1}{p_-}\dot{x}^A x'^A + i\bar{\theta}^I\Gamma^-\partial_1\theta^I \ . \qquad (4.21) \end{aligned}$$

By substituting back into (4.15), and performing the analogous operation for the x'^- variation, these leading-order solutions provide the following expansions for the objects that enter into the x^- equation of motion:

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \dot{x}^-} &= \frac{1}{2} h^{00} \left\{ \frac{2p_-}{R^2} + \frac{1}{R^4} \left[p_-(z^2 - y^2) - \frac{1}{p_-} \left[(\dot{x}^A)^2 + (x'^A)^2 \right] + i \bar{\theta}^I \Gamma^- \partial_0 \theta^I \right] \right\} \\
&\quad + \frac{i}{2R^4} s^{IJ} \bar{\theta}^I \Gamma^- \partial_1 \theta^J + \mathcal{O}(R^{-6}) \\
\frac{\delta \mathcal{L}}{\delta x'^-} &= \frac{h^{01} p_-}{R^2} + \frac{h^{11}}{R^4} \left(-\frac{1}{p_-} \dot{x}^A x'^A + \frac{i}{2} \bar{\theta}^I \Gamma^- \partial_1 \theta^I \right) - \frac{i}{2R^4} s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J + \mathcal{O}(R^{-6}) \ .
\end{aligned}
\tag{4.22}
$$

It is obvious from these expressions that the x^- equation of motion will not be consistent with the standard choice of flat worldsheet metric ($h^{00} = -h^{11} = 1, h^{01} = 0$). We therefore expand h^{ab} in powers of R^{-1}, taking it to be flat at leading order and allowing the higher-order terms (the $\tilde{h}^{ab}$) to depend on the physical variables in some way:

$$
h^{00} = -1 + \frac{\tilde{h}^{00}}{R^2} + \mathcal{O}(R^{-4}) \qquad h^{11} = 1 + \frac{\tilde{h}^{11}}{R^2} + \mathcal{O}(R^{-4}) \qquad h^{01} = \frac{\tilde{h}^{01}}{R^2} + \mathcal{O}(R^{-4}) \ . \tag{4.23}
$$

Using (4.20) and (4.21), we find that the specific metric choice

$$
\tilde{h}^{00} = \frac{1}{2}(z^2 - y^2) - \frac{1}{2p_-^2} \left[(\dot{x}^A)^2 + (x'^A)^2 \right] + \frac{i}{2p_-} \bar{\theta}^I \Gamma^- \partial_0 \theta^I - \frac{i}{2p_-} s^{IJ} \bar{\theta}^I \Gamma^- \partial_1 \theta^J \tag{4.24}
$$

$$
\tilde{h}^{01} = \frac{1}{p_-^2} \dot{x}^A x'^A - \frac{i}{2p_-} \bar{\theta}^I \Gamma^- \partial_1 \theta^I + \frac{i}{2p_-} s^{IJ} \bar{\theta}^I \Gamma^- \partial_0 \theta^J \tag{4.25}
$$

simplifies the expressions of (4.22) to

$$
\frac{\delta \mathcal{L}}{\delta \dot{x}^-} = 1 + \mathcal{O}(R^{-4}) \qquad \frac{\delta \mathcal{L}}{\delta x'^-} = \mathcal{O}(R^{-4}) \ . \tag{4.26}
$$

The x^- equation of motion is then consistent with the standard light-cone gauge choice $\dot{x}^+ = p_-$ to $\mathcal{O}(1/R^2)$ (with no corrections to p_-, which must remain constant). Note that $\tilde{h}^{00} = -\tilde{h}_{00}$ and $\tilde{h}_{00} = \tilde{h}_{11}$. The fact that these curvature corrections have bi-fermionic contributions is ultimately due to the presence of a non-vanishing G_{--} term in the expanded metric (3.35).

Since the worldsheet metric is known to $\mathcal{O}(1/R^2)$, x^- can now be determined to this order from the covariant gauge constraints (4.16). By invoking the leading-order solutions (4.17,4.18), we can simplify the equations to some extent:

$$
T_{00} = \frac{1}{2} \left(L_0^\mu L_0^\mu + L_1^\mu L_1^\mu \right) + \frac{\tilde{h}^{00}}{R^2} L_1^\mu L_1^\mu + \mathcal{O}(R^{-3}) = 0 \tag{4.27}
$$

$$
T_{01} = L_0^\mu L_1^\mu - \frac{\tilde{h}_{01}}{R^2} L_1^\mu L_1^\mu + \mathcal{O}(R^{-3}) = 0 \ . \tag{4.28}
$$

Equation (4.27) may be expanded to solve for a_2, the first subleading correction to $\dot{x}^-$:

$$
\begin{aligned}
T_{00} \;=\;& 2p_-a_2 + a_0^2 - 2p_-y^2a_0 + a_0'^2 + \frac{1}{2}(\dot{z}^2z^2 - \dot{y}^2y^2) + \frac{p_-^2}{2}(y^4 - z^4) + \frac{1}{2}(z'^2z^2 - y'^2y^2) \\
& +(z^2 - y^2)(x'^A)^2 - \frac{1}{p_-^2}\left[(\dot{x}^A)^2 + (x'^A)^2\right](x'^A)^2 + \frac{i}{p_-}(x'^A)^2\bar{\theta}^I\Gamma^-\partial_0\theta^I \\
& -\frac{i}{p_-}(x'^A)^2s^{IJ}\bar{\theta}^I\Gamma^-\partial_1\theta^J - ia_0\Delta_0^- - 2ip_-\Delta_2^- - 2ip_-(\Delta_0^-)_{\theta^4} + \frac{ip_-}{2}(y^2 - z^2)\Delta_0^- \\
& -2i(\dot{x}^A\Delta_1^A + x'^A\Delta_1^A) - ia_0'\Delta_0'^- = 0\ . \qquad (4.29)
\end{aligned}
$$

The remaining independent component T_{01} is the current associated with translation symmetry on the closed-string worldsheet. Enforcing the constraint $T_{01} = 0$ is equivalent to imposing the level-matching condition on physical string states. This condition can be used to fix higher-order corrections to x'^-, as is required by conformal invariance on the worldsheet. However, since our goal is to examine curvature corrections to the pp-wave limit using first-order perturbation theory, we will only need to enforce the level-matching condition on string states that are eigenstates of the pp-wave theory. We therefore need only consider the equation $T_{01} = 0$ to leading order in the expansion, which yields (4.21) above. If we were interested in physical eigenstates of the geometry corrected to $\mathcal{O}(1/R^2)$ (ie. solving the theory exactly to this order), we would be forced to solve $T_{01} = 0$ to $\mathcal{O}(1/R^2)$.

With solutions to the x^- equations of motion and an expansion of the worldsheet metric to the order of interest, we may proceed with expressing the Hamiltonian as the generator of light-cone time translation: $p_+ = \delta\mathcal{L}/\delta\dot{x}^+$. It is helpful to first vary Δ^μ with respect to $\partial_0 t$ and $\partial_0\phi$:

$$
\begin{aligned}
\frac{\delta\Delta^\mu}{\delta(\partial_0 t)} &= \bar{\theta}^I\Gamma^\mu\left[-\frac{1}{2R^3}z_j\Gamma^{0j}\theta^I - \frac{1}{2}\epsilon^{IJ}\Pi\left(\frac{1}{R^2} + \frac{z^2}{2R^4}\right)\theta^J\right] + \mathcal{O}(R^{-6}) \qquad (4.30)\\
\frac{\delta\Delta^\mu}{\delta(\partial_0 \phi)} &= \bar{\theta}^I\Gamma^\mu\left[-\frac{1}{2R^3}y_{j'}\Gamma^{9j'}\theta^I - \frac{1}{2}\epsilon^{IJ}\Pi\left(\frac{1}{R^2} - \frac{y^2}{2R^4}\right)\theta^J\right] + \mathcal{O}(R^{-6})\ . \qquad (4.31)
\end{aligned}
$$

The kinetic term in the Lagrangian (3.11) yields

$$
\begin{aligned}
\frac{\delta \mathcal{L}_{\rm Kin}}{\delta \dot{x}^+} &= \frac{1}{R^2}\left\{p_-(x^A)^2 - \dot{x}^- + i\Delta_0^- - ip_-\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J\right\} \\
&+\frac{1}{R^4}\bigg\{-\frac{p_-}{2}(y^4-z^4) + y^2\dot{x}^- + i\Delta_2^- + i(\Delta_0^-)_{\theta^4} + \frac{i}{4}(z^2-y^2)\Delta_0^- \\
&-\frac{ip_-}{2}(z^2-y^2)\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J - \frac{ip_-}{12}\bar\theta^I\Gamma^-(\mathcal{M}^2)^{IJ}\epsilon^{JL}\Pi\theta^L \\
&+\frac{i}{4}\dot{x}^A\bar\theta^I\Gamma^A\left(z_k\Gamma^{-k} - y_{k'}\Gamma^{-k'}\right)\theta^I - \frac{i}{2}(\dot{x}^-)\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J + \bigg[-\frac{1}{2}(z^2-y^2) \\
&+\frac{1}{2p_-^2}\left[(\dot{x}^A)^2 + (x'^A)^2\right] - \frac{i}{2p_-}\bar\theta^I\Gamma^-\partial_0\theta^I + \frac{i}{2p_-}s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J\bigg]\bigg[p_-(x^A)^2 - \dot{x}^- \\
&+i\Delta_0^- - ip_-\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J\bigg] + \bigg[\frac{1}{p_-^2}\dot{x}^A x'^A - \frac{i}{2p_-}\bar\theta^I\Gamma^-\partial_1\theta^I \\
&+\frac{i}{2p_-}s^{IJ}\bar\theta^I\Gamma^-\partial_0\theta^J\bigg]\left(x'^- - i\Delta'^-_0\right)\bigg\} + \mathcal{O}(R^{-6})\ ,
\end{aligned}
\tag{4.32}
$$

while the Wess-Zumino term (3.12) gives

$$
\begin{aligned}
\frac{\delta \mathcal{L}_{\rm WZ}}{\delta \dot{x}^+} &= \frac{i}{R^2}s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J + \frac{1}{R^4}\bigg\{\frac{i}{4}s^{IJ}\bar\theta^I\Gamma^-(z'_j z_k\Gamma^{jk} - y'_{j'}y_{k'})\theta^J + \frac{i}{12}s^{IJ}\bar\theta^I\Gamma^-(\mathcal{M}^2)^{JL}\partial_1\theta^L \\
&-\frac{i}{4}(y^2-z^2)s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J + \frac{i}{4}x'^A s^{IJ}\bar\theta^I\Gamma^A(y_{j'}\Gamma^{-j'} - z_j\Gamma^{-j})\theta^J\bigg\} + \mathcal{O}(R^{-6})\ .
\end{aligned}
\tag{4.33}
$$

The variation is completed prior to any gauge fixing (with the worldsheet metric held fixed). After computing the variation, the light-cone coordinates $x^\pm$ and the worldsheet metric corrections $\tilde h^{00}, \tilde h^{01}$ are to be replaced with dynamical variables according to the x^- equations of motion and the gauge conditions $x^+ = \tau$ and $T_{ab} = 0$. Hence, using a_0 and a_2 determined from the covariant gauge constraints (4.20,4.29), we remove x^- (x^+ has already been replaced with $p_-\tau$ in the above variations) and restore proper powers of R in the vielbeins (so that the desired corrections enter at $\mathcal{O}(1/R^2)$). As expected, the pp-wave Hamiltonian emerges at leading order:

$$
\mathcal{H}_{pp} = \frac{p_-}{2}(x^A)^2 + \frac{1}{2p_-}\left[(\dot{x}^A)^2 + (x'^A)^2\right] - ip_-\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J + is^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J\ . \tag{4.34}
$$

The first curvature correction to the pp-wave limit is found to be

$$
\begin{aligned}
\mathcal{H}_{\rm int} \;=\; & \frac{1}{R^2}\Bigg\{ \frac{1}{4p_-}\left[y^2(\dot{z}^2 - z'^2 - 2y'^2) + z^2(-\dot{y}^2 + y'^2 + 2z'^2)\right] \\
& + \frac{1}{8p_-^3}\left[3(\dot{x}^A)^2 - (x'^A)^2\right]\left[(\dot{x}^A)^2 + (x'^A)^2\right] + \frac{p_-}{8}\left[(x^A)^2\right]^2 - \frac{1}{2p_-^3}(\dot{x}^A x'^A)^2 \\
& - \frac{i}{4p_-}\sum_{a=0}^{1}\bar\theta^I(\partial_a x^A\Gamma^A)\epsilon^{IJ}\Gamma^-\Pi(\partial_a x^B\Gamma^B)\theta^J - \frac{i}{2}p_-(x^A)^2\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J \\
& - \frac{i}{2p_-^2}(\dot{x}^A)^2\bar\theta^I\Gamma^-\partial_0\theta^I - \frac{ip_-}{12}\bar\theta^I\Gamma^-(\mathcal{M}^2)^{IJ}\epsilon^{JL}\Pi\theta^L - \frac{p_-}{2}(\bar\theta^I\Gamma^-\epsilon^{IJ}\Pi\theta^J)^2 \\
& - \frac{i}{2p_-^2}(\dot{x}^A x'^A)s^{IJ}\bar\theta^I\Gamma^-\partial_0\theta^J - \frac{i}{4}(y^2 - z^2)s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J \\
& + \frac{i}{4}x'^A s^{IJ}\bar\theta^I\Gamma^A(y_{j'}\Gamma^{-j'} - z_j\Gamma^{-j})\theta^J + \frac{i}{4}s^{IJ}\bar\theta^I\Gamma^-(z'_j z_k\Gamma^{jk} - y'_{j'}y_{k'}\Gamma^{j'k'})\theta^J \\
& + \frac{i}{4p_-^2}\left[(\dot{x}^A)^2 - (x'^A)^2\right]s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J + \frac{i}{12}s^{IJ}\bar\theta^I\Gamma^-(\mathcal{M}^2)^{JL}\partial_1\theta^L \\
& + \frac{1}{2}(s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J)(\bar\theta^K\Gamma^-\epsilon^{KL}\Pi\theta^L) + \frac{i}{4}(x^A)^2 s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J\Bigg\}\;. \qquad (4.35)
\end{aligned}
$$

The full Lagrangian (3.11,3.12) can also be expressed to this order. In terms of the quantities found in equations (4.12,4.13,4.14,4.24,4.25), the kinetic term $\mathcal{L}_{\rm Kin} = -\frac{1}{2}h^{ab}L_a^\mu L_b^\mu$ can be written schematically as

$$
\begin{aligned}
\mathcal{L}_{\rm Kin} \;=\; & \frac{1}{2}\left(L_0^\mu L_0^\mu - L_1^\mu L_1^\mu\right)_2 + \frac{1}{2R^2}\left(L_0^\mu L_0^\mu - L_1^\mu L_1^\mu\right)_4 - \frac{1}{2R^2}\tilde{h}^{00}\left(L_0^\mu L_0^\mu\right)_2 \\
& + \frac{1}{2R^2}\tilde{h}^{00}\left(L_1^\mu L_1^\mu\right)_2 - \frac{1}{R^2}\tilde{h}^{01}\left(L_0^\mu L_1^\mu\right)_2 + \mathcal{O}(R^{-4})\;, \qquad (4.36)
\end{aligned}
$$

where external subscripts indicate quadratic or quartic order in fields. The Wess-Zumino term is given explicitly by:

$$
\begin{aligned}
\mathcal{L}_{\rm WZ} \;=\; & -2i\epsilon^{ab}\int_0^1 dt L_{at}^\mu s^{IJ}\bar\theta^I\Gamma^\mu L_{bt}^J \\
\approx\; & -ip_-\left(s^{IJ}\bar\theta^I\Gamma^-\partial_1\theta^J\right) - \frac{i}{R^2}\Bigg\{p_-\Box'^-_2 + p_-(\Box'^-_0)_{\theta^4} + \frac{p_-}{4}(z^2 - y^2)\Box'^-_0 + \frac{1}{2}\dot{x}^-\Box'^-_0 \\
& - \frac{1}{2}x'^-\Box^-_0 + \dot{x}^A\Box'^A_1 - x'^A\Box^A_1\Bigg\} + \mathcal{O}(R^{-4})\;. \qquad (4.37)
\end{aligned}
$$

It will be useful to recast both the Hamiltonian and Lagrangian in 16-component notation

(details may be found in Appendix A):

$$
\begin{aligned}
\mathcal{H} \;=\;& \frac{1}{2p_-}\Big((\dot{x}^A)^2 + (x'^A)^2 + p_-^2 (x^A)^2\Big) - p_-\psi^\dagger\Pi\psi + \frac{i}{2}(\psi\psi' + \psi^\dagger\psi'^\dagger) \\
&+\frac{1}{R^2}\Bigg\{\frac{z^2}{4p_-}\Big[y'^2 + 2z'^2 - \dot{y}^2\Big] - \frac{y^2}{4p_-}\Big[z'^2 + 2y'^2 - \dot{z}^2\Big] - \frac{1}{2p_-^3}(\dot{x}^A x'^A)^2 \\
&+\frac{1}{8p_-^3}\Big[3(\dot{x}^A)^2 - (x'^A)^2\Big]\Big[(\dot{x}^A)^2 + (x'^A)^2\Big] + \frac{p_-}{8}\big[(x^A)^2\big]^2 \\
&+\frac{i}{8}\psi\Big(z_k z_j'\gamma^{jk} - y_{k'}y_{j'}'\gamma^{j'k'} + x'^A(z_k\bar{\gamma}^A\gamma^k - y_{k'}\bar{\gamma}^A\gamma^{k'})\Big)\psi - \frac{i}{4p_-^2}(\dot{x}^A)^2\Big[\psi\dot{\psi}^\dagger + \psi^\dagger\dot{\psi}\Big] \\
&+\frac{i}{8}\psi^\dagger\Big(z_k z_j'\gamma^{jk} - y_{k'}y_{j'}'\gamma^{j'k'} + x'^A(z_k\bar{\gamma}^A\gamma^k - y_{k'}\bar{\gamma}^A\gamma^{k'})\Big)\psi^\dagger \\
&+\frac{1}{2p_-}\Big(\dot{z}^i\dot{y}^{j'} + z'^i y'^{j'}\Big)\psi^\dagger\gamma^{ij'}\Pi\psi + \frac{i}{8}(z^2 - y^2)(\psi\psi' + \psi^\dagger\psi'^\dagger) \\
&-\frac{1}{4p_-}\Big[(\dot{z}^2 - \dot{y}^2) + (z'^2 - y'^2)\Big]\psi^\dagger\Pi\psi + \frac{i}{8}\left[\frac{1}{p_-^2}\Big((\dot{x}^A)^2 - (x'^A)^2\Big) + (x^A)^2\right](\psi\psi' + \psi^\dagger\psi'^\dagger) \\
&-\frac{p_-}{2}(x^A)^2(\psi^\dagger\Pi\psi) - \frac{i}{4p_-^2}(\dot{x}^A x'^A)(\psi\dot{\psi} + \psi^\dagger\dot{\psi}^\dagger) + \frac{p_-}{48}(\psi^\dagger\gamma^{jk}\psi)(\psi^\dagger\gamma^{jk}\psi) \\
&-\frac{p_-}{48}(\psi^\dagger\gamma^{j'k'}\psi)(\psi^\dagger\gamma^{j'k'}\psi) - \frac{i}{192}(\psi\gamma^{jk}\psi + \psi^\dagger\gamma^{jk}\psi^\dagger)(\psi^\dagger\gamma^{jk}\Pi\psi' - \psi\gamma^{jk}\Pi\psi'^\dagger) \\
&+\frac{p_-}{2}(\psi^\dagger\Pi\psi)(\psi^\dagger\Pi\psi) + \frac{i}{192}(\psi\gamma^{j'k'}\psi + \psi^\dagger\gamma^{j'k'}\psi^\dagger)(\psi^\dagger\gamma^{j'k'}\Pi\psi' - \psi\gamma^{j'k'}\Pi\psi'^\dagger) \\
&-\frac{i}{4}(\psi\psi' + \psi^\dagger\psi'^\dagger)(\psi^\dagger\Pi\psi)\Bigg\} + \mathcal{O}(R^{-4})\ . \qquad (4.38)
\end{aligned}
$$

One could scale the length of the worldsheet such that all p_- are absorbed into the upper limit on worldsheet integration over $d\sigma$. To organize correction terms by their corresponding coupling strength in the gauge theory, however, we find it convenient to keep factors of p_-

explicit in the above expression. The Lagrangian can be computed from (4.36,4.37), giving

$$
\begin{aligned}
\mathcal{L}_{\text{Kin}} \;=\;& p_-\dot{x}^- - \frac{1}{2}\left[p_-^2(x^A)^2 - (\dot{x}^A)^2 + (x'^A)^2\right] - \frac{ip_-}{2}(\psi\dot{\psi}^\dagger + \psi^\dagger\dot{\psi}) - p_-^2\psi\Pi\psi^\dagger \\
&+\frac{1}{2R^2}\Big\{(\dot{x}^-)^2 - 2p_-y^2\dot{x}^- + \frac{1}{2}(\dot{z}^2z^2 - \dot{y}^2y^2) + \frac{p_-^2}{2}(y^4 - z^4) \\
&-\frac{ip_-}{4}(\dot{z}_jz_k)(\psi\gamma^{jk}\psi^\dagger + \psi^\dagger\gamma^{jk}\psi) + \frac{ip_-}{4}(\dot{y}_{j'}y_{k'})(\psi\gamma^{j'k'}\psi^\dagger + \psi^\dagger\gamma^{j'k'}\psi) \\
&-\frac{ip_-}{48}(\psi\gamma^{jk}\psi^\dagger)(\psi\gamma^{jk}\Pi\dot{\psi}^\dagger - \psi^\dagger\gamma^{jk}\Pi\dot{\psi}) + \frac{ip_-}{48}(\psi\gamma^{j'k'}\psi^\dagger)(\psi\gamma^{j'k'}\Pi\dot{\psi}^\dagger - \psi^\dagger\gamma^{j'k'}\Pi\dot{\psi}) \\
&+\frac{i}{2}\left[\frac{p_-}{2}(y^2 - z^2) - \dot{x}^-\right](\psi\dot{\psi}^\dagger + \psi^\dagger\dot{\psi}) - p_-\left[2\dot{x}^- - p_-(y^2 - z^2)\right]\psi\Pi\psi^\dagger \\
&-\frac{p_-^2}{24}(\psi^\dagger\gamma^{jk}\psi)^2 + \frac{p_-^2}{24}(\psi^\dagger\gamma^{j'k'}\psi)^2 + \frac{ip_-}{4}(\dot{x}^Az_j)(\psi\gamma^A\bar{\gamma}^j\psi^\dagger + \psi^\dagger\gamma^A\bar{\gamma}^j\psi) \\
&-\frac{ip_-}{4}(\dot{x}^Ay_{j'})(\psi\gamma^A\bar{\gamma}^{j'}\psi^\dagger + \psi^\dagger\gamma^A\bar{\gamma}^{j'}\psi) + \frac{1}{4}(\dot{x}^A\dot{x}^B)(\psi^\dagger\gamma^A\Pi\bar{\gamma}^B\psi - \psi\gamma^A\Pi\bar{\gamma}^B\psi^\dagger) \\
&-\frac{1}{2}(z'^2z^2 - y'^2y^2) - (x'^-)^2 + \frac{i}{2}x'^-(\psi\psi'^\dagger + \psi^\dagger\psi') \\
&-\frac{1}{4}(x'^Ax'^B)(\psi^\dagger\gamma^A\Pi\bar{\gamma}^B\psi - \psi\gamma^A\Pi\bar{\gamma}^B\psi^\dagger) - \tilde{h}^{00}\big[2p_-\dot{x}^- - p_-^2(x^A)^2 + (\dot{x}^A)^2 - (x'^A)^2 \\
&-ip_-(\psi\dot{\psi}^\dagger + \psi^\dagger\dot{\psi}) - 2p_-^2\psi\Pi\psi^\dagger\big] - 2\tilde{h}^{01}\left[p_-x'^- + \dot{x}^Ax'^A - \frac{ip_-}{2}(\psi\psi'^\dagger + \psi^\dagger\psi')\right]\Big\} \\
&+\mathcal{O}(R^{-4})\ , \qquad (4.39)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\text{WZ}} \;=\;& -\frac{ip_-}{2}(\psi\psi' + \psi^\dagger\psi'^\dagger) - \frac{i}{R^2}\Big\{\frac{p_-}{8}(z'_jz_k)(\psi\gamma^{jk}\psi + \psi^\dagger\gamma^{jk}\psi^\dagger) \\
&-\frac{p_-}{8}(y'_{j'}y_{k'})(\psi\gamma^{j'k'}\psi + \psi^\dagger\gamma^{j'k'}\psi^\dagger) + \frac{1}{4}\left[\dot{x}^- + \frac{p_-}{2}(z^2 - y^2)\right](\psi\psi' + \psi^\dagger\psi'^\dagger) \\
&-\frac{1}{4}(x'^-)(\psi\dot{\psi} + \psi^\dagger\dot{\psi}^\dagger) + \frac{i}{8}(x'^A\dot{x}^B + \dot{x}^Ax'^B)(\psi^\dagger\gamma^A\Pi\bar{\gamma}^B\psi^\dagger - \psi\gamma^A\Pi\bar{\gamma}^B\psi) \\
&+\frac{p_-}{8}(x'^Az_j)(\psi^\dagger\gamma^A\bar{\gamma}^j\psi^\dagger + \psi\gamma^A\bar{\gamma}^j\psi) - \frac{p_-}{8}(x'^Ay_{j'})(\psi^\dagger\gamma^A\bar{\gamma}^{j'}\psi^\dagger + \psi\gamma^A\bar{\gamma}^{j'}\psi) \\
&+\frac{p_-}{8}(\psi\gamma^{jk}\psi + \psi^\dagger\gamma^{jk}\psi^\dagger)(\psi\gamma^{jk}\Pi\psi'^\dagger - \psi^\dagger\gamma^{jk}\Pi\psi') \\
&-\frac{p_-}{8}(\psi\gamma^{j'k'}\psi + \psi^\dagger\gamma^{j'k'}\psi^\dagger)(\psi\gamma^{j'k'}\Pi\psi'^\dagger - \psi^\dagger\gamma^{j'k'}\Pi\psi')\Big\} + \mathcal{O}(R^{-4})\ . \qquad (4.40)
\end{aligned}
$$

For later convenience, the Lagrangian is not fully gauge fixed, though we set $\dot{x}^+$ to p_- for simplicity and ignore any $\ddot{x}^+$ that arise through partial integration (since we will ultimately choose the light-cone gauge $x^+ = p_-\tau$). As noted above, sending $h^{00} \to -1 + \tilde{h}^{00}/R^2$ simply rewrites the function h^{00}, and does not amount to a particular gauge choice for the worldsheet metric.

5 Quantization of the lightcone gauge Hamiltonian

Our goal is to calculate explicit energy corrections due to the rather complicated perturbed Hamiltonian derived in the last section. To explain our strategy, we begin with a review of the pp-wave energy spectrum in the Penrose limit. This limit is obtained by keeping only the leading term in R^{-1} in the Hamiltonian expansion of (4.38) and leads to linear equations of motion for the fields. The eight bosonic transverse string coordinates obey the equation

$$\ddot{x}^A - x''^A + p_-^2 x^A = 0 \ . \tag{5.1}$$

This is solved by the usual expansion in terms of Fourier modes

$$\begin{aligned} x^A(\sigma,\tau) &= \sum_{n=-\infty}^{\infty} x_n^A(\tau) e^{-ik_n\sigma} \\ x_n^A(\tau) &= \frac{i}{\sqrt{2\omega_n}}(a_n^A e^{-i\omega_n\tau} - a_{-n}^{A\dagger} e^{i\omega_n\tau}) \ , \end{aligned} \tag{5.2}$$

where $k_n = n$ (integer), $\omega_n = \sqrt{p_-^2 + k_n^2}$, and the raising and lowering operators obey the commutation relation $[a_m^A, a_n^{B\dagger}] = \delta_{mn}\delta^{AB}$. The bosonic piece of the pp-wave Hamiltonian takes the form

$$\mathcal{H}_{\rm pp}^B = \frac{1}{p_-}\sum_{n=-\infty}^{\infty} \omega_n \left(a_n^{A\dagger} a_n^A + 4 \right) \ . \tag{5.3}$$

The fermionic equations of motion are

$$(\dot{\psi}^\dagger + \psi') + ip_-\Pi\psi^\dagger = 0 \tag{5.4}$$
$$(\dot{\psi} + \psi'^\dagger) - ip_-\Pi\psi = 0 \ , \tag{5.5}$$

where ψ is a 16-component complex SO(9,1) Weyl spinor. As mentioned earlier, ψ is further restricted by a light-cone gauge fixing condition $\bar{\gamma}^9\psi = \psi$ which reduces the number of spinor components to 8 (details are given in the Appendix). In what follows, ψ, and the various matrices acting on it, should therefore be regarded as 8-dimensional. The fermionic equations of motion are solved by

$$\psi = \sum_{n=-\infty}^{\infty} \psi_n(\tau) e^{-ik_n\sigma} \tag{5.6}$$
$$\psi_n(\tau) = \frac{1}{2\sqrt{p_-}}\left(A_n b_n e^{-i\omega_n\tau} + B_n b_{-n}^\dagger e^{i\omega_n\tau} \right) e^{-ik_n\sigma} \tag{5.7}$$
$$\psi_n^\dagger(\tau) = \frac{1}{2\sqrt{p_-}}\left(\Pi B_n b_n e^{-i\omega_n\tau} - \Pi A_n b_{-n}^\dagger e^{i\omega_n\tau} \right) e^{-ik_n\sigma} \ , \tag{5.8}$$

where we have defined

$$A_n \equiv \frac{1}{\sqrt{\omega_n}}\left(\sqrt{\omega_n - k_n} - \sqrt{\omega_n + k_n}\Pi\right) \tag{5.9}$$

$$B_n \equiv \frac{1}{\sqrt{\omega_n}}\left(\sqrt{\omega_n + k_n} + \sqrt{\omega_n - k_n}\Pi\right) . \tag{5.10}$$

The anticommuting mode operators $b_n, b_n^\dagger$ carry a spinor index which takes 8 values. In the gamma matrix representation described in the Appendix, the matrix Π is diagonal and assigns eigenvalues ± 1 to the mode operators. The fermionic canonical momentum is $\rho = ip_-\psi^\dagger$, which implies that the fermionic creation and annihilation operators obey the anticommutation rule $\{b_m^\alpha, b_n^{\beta\dagger}\} = \delta^{\alpha\beta}\delta_{mn}$. The fermionic piece of the pp-wave Hamiltonian can be written in terms of these operators as

$$\mathcal{H}_{\rm pp}^F = \frac{1}{p_-}\sum_{n=-\infty}^{\infty}\omega_n\left(b_n^{\alpha\dagger}b_n^\alpha - 4\right) . \tag{5.11}$$

Given our earlier conventions, it is necessary to invoke the coordinate reflection $x^\mu \to -x^\mu$ (Metsaev studied a similar operation on the pp-wave Hamiltonian in [2]). Such a transformation is, at this stage, equivalent to sending $x^A \to -x^A$, $p_- \to -p_-$, and $\mathcal{H} \to -\mathcal{H}$. In essence, this operation allows us to choose the positive-energy solutions to the fermionic equations of motion while maintaining our convention that $b^{\alpha^\dagger}$ represent a creation operator and b^α denote an annihilation operator. The total pp-wave Hamiltonian

$$\mathcal{H}_{\rm pp} = \frac{1}{p_-}\sum_{n=-\infty}^{\infty}\omega_n\left(a_n^{A\dagger}a_n^A + b_n^{\alpha\dagger}b_n^\alpha\right) \tag{5.12}$$

is just a collection of free, equal mass fermionic and bosonic oscillators.

Canonical quantization requires that we express the Hamiltonian in terms of physical variables and conjugate momenta. At leading order in $1/R^2$, $\dot{x}^A$ is canonically conjugate to x^A and can be expanded in terms of creation and annihilation operators. Beyond leading order, however, the conjugate variable $p_A = \delta\mathcal{L}/\delta\dot{x}^A$ differs from $\dot{x}^A$ by terms of $\mathcal{O}(1/R^2)$. Substituting these $\mathcal{O}(1/R^2)$ corrected expressions for canonical momenta into the pp-wave Hamiltonian

$$\mathcal{H}_{\rm pp} \sim (\dot{x}^A)^2 + \psi^\dagger\Pi\psi + \psi^\dagger\psi'^\dagger \tag{5.13}$$

to express it as a function of canonical variables will yield indirect $\mathcal{O}(1/R^2)$ corrections to the Hamiltonian (to which we must add the contribution of explicit $\mathcal{O}(1/R^2)$ corrections to the action). For example, bosonic momenta in the $SO(4)$ descending from the AdS_5 subspace

take the following corrections:

$$
\begin{aligned}
p_k &= \dot{z}_k + \frac{1}{R^2}\Bigg\{ \frac{1}{2} y^2 p_k + \frac{1}{2p_-^2}\left[(p_A)^2 + (x'^A)^2\right] p_k - \frac{1}{p_-^2}(p_A x'^A) z'_k - \frac{i}{2p_-} p_k \bar\theta^I \Gamma^- \partial_0 \theta^I \\
&\quad + \frac{i}{2p_-} p_k s^{IJ} \bar\theta^I \Gamma^- \partial_1 \theta^J - \frac{ip_-}{4} \bar\theta^I \Gamma^- z_j \Gamma_k{}^j \theta^I - \frac{ip_-}{4} \bar\theta^I \Gamma^k \left(z_j \Gamma^{-j} - y_{j'} \Gamma^{-j'} \right) \theta^I \\
&\quad + \frac{i}{4} p_A \epsilon^{IJ} \bar\theta^I \Gamma^- \left(\Gamma_k \Pi \Gamma^A + \Gamma^A \Pi \Gamma_k \right) \theta^J + \frac{i}{2p_-} z'_k \bar\theta^I \Gamma^- \partial_1 \theta^I - \frac{i}{2p_-} z'_k s^{IJ} \bar\theta^I \Gamma^- \partial_0 \theta^J \\
&\quad + \frac{i}{4} x'^A s^{IJ} \epsilon^{JK} \bar\theta^I \Gamma^- \left(\Gamma_k \Pi \Gamma^A - \Gamma^A \Pi \Gamma_k \right) \theta^K \Bigg\} + \mathcal{O}(R^{-4}) \, . \qquad (5.14)
\end{aligned}
$$

The leading-order relationship $p_k = \dot{z}_k$ has been substituted into the correction term at $\mathcal{O}(1/R^2)$, and the light-cone gauge choice $x^+ = p_-\tau$ has been fixed after the variation.

To compute fermionic momenta $\rho = \delta\mathcal{L}/\delta\dot\psi$, it is convenient to work with complex 16-component spinors. Terms in $\mathcal{L}$ relevant to the fermionic momenta ρ are as follows:

$$
\begin{aligned}
\mathcal{L} \ &\sim \ -ip_- \left(\psi^\dagger \dot\psi \right) - \frac{i}{R^2} \Bigg\{ \frac{1}{4} \left[\dot{x}^- + \frac{p_-}{2}(z^2 - y^2) \right] \left(\psi \dot\psi^\dagger + \psi^\dagger \dot\psi \right) - \frac{p_- \tilde{h}^{00}}{2} \left(\psi \dot\psi^\dagger + \psi^\dagger \dot\psi \right) \\
&+ \frac{p_-}{96} \left(\psi \gamma^{jk} \psi^\dagger \right) \left(\psi \gamma^{jk} \Pi \dot\psi^\dagger - \psi^\dagger \gamma^{jk} \Pi \dot\psi \right) - \frac{x'^-}{4} \left(\psi \dot\psi + \psi^\dagger \dot\psi^\dagger \right) - (j,k \rightleftharpoons j',k') \Bigg\} + \mathcal{O}(R^{-4}) \, . \qquad (5.15)
\end{aligned}
$$

This structure can be manipulated to simplify the subsequent calculation. Using partial integration, we can make the following replacement at leading order:

$$
\frac{ip_-}{2} \left(\psi^\dagger \dot\psi + \psi \dot\psi^\dagger \right) = ip_- \left(\psi^\dagger \dot\psi \right) + \text{surface terms.} \qquad (5.16)
$$

Operations of this sort have no effect on the x^- equation of motion or the preceding calculation of $\delta\mathcal{L}/\delta\dot{x}^+$, for example. Similarly, terms in $\mathcal{L}$ containing the matrix $(\mathcal{M})^2$ may be transformed according to

$$
-\frac{ip_-}{96} \left(\psi \gamma^{jk} \psi^\dagger \right) \left(\psi \gamma^{jk} \Pi \dot\psi^\dagger - \psi^\dagger \gamma^{jk} \Pi \dot\psi \right) \ = \ \frac{ip_-}{48} \left(\psi \gamma^{jk} \psi^\dagger \right) \left(\psi^\dagger \gamma^{jk} \Pi \dot\psi \right) \, . \qquad (5.17)
$$

Terms of the form

$$
\frac{1}{4} \left(\dot{x}^- \right) \left(\psi \dot\psi^\dagger + \psi^\dagger \dot\psi \right) \, , \qquad (5.18)
$$

however, cannot be treated in the same manner. The presence of (5.18) ultimately imposes a set of second-class constraints on the theory, and we will eventually be lead to treat $\psi^\dagger$ as a constrained, dynamical degree of freedom in the Lagrangian. The fermionic momenta

therefore take the form

$$
\begin{aligned}
\rho_\alpha &= ip_-\psi^\dagger_\alpha + \frac{1}{R^2}\Bigg\{\frac{i}{4}\left(\dot{x}^- + \frac{p_-}{2}(z^2-y^2)\right)\psi^\dagger_\alpha - \frac{ip_-}{2}\tilde{h}^{00}\psi^\dagger_\alpha - \frac{ix'^-}{4}\psi_\alpha \\
&\quad -\frac{ip_-}{48}\left[\left(\psi\gamma^{jk}\psi^\dagger\right)\left(\psi^\dagger\gamma^{jk}\Pi\right)_\alpha - (j,k \rightleftharpoons j',k')\right]\Bigg\} + \mathcal{O}(R^{-4}) \qquad (5.19)\\
\rho^\dagger_\alpha &= \frac{1}{R^2}\Bigg\{\frac{i}{4}\left(\dot{x}^- + \frac{p_-}{2}(z^2-y^2)\right)\psi_\alpha - \frac{ip_-}{2}\tilde{h}^{00}\psi_\alpha - \frac{ix'^-}{4}\psi^\dagger_\alpha\Bigg\} + \mathcal{O}(R^{-4})\ . \qquad (5.20)
\end{aligned}
$$

Using (4.20) and (4.21) to replace $\dot{x}^-$ and x'^- at leading order (in 16-component spinor notation), and using (4.24) to implement the appropriate curvature corrections to the h^{00} component of the worldsheet metric, we find

$$
\begin{aligned}
\rho &= ip_-\psi^\dagger + \frac{1}{R^2}\Bigg\{\frac{1}{4}y^2\rho + \frac{1}{8p_-^2}\left[(p_A^2) + (x'^A)^2\right]\rho + \frac{i}{4p_-}(p_A x'^A)\psi + \frac{i}{4p_-}\left(\rho\Pi\psi\right)\rho \\
&\quad -\frac{i}{8p_-}\left(\psi\rho' + \rho\psi'\right)\psi + \frac{i}{8p_-}\left(\psi\psi' - \frac{1}{p_-^2}\rho\rho'\right)\rho \\
&\quad +\frac{i}{48p_-}\left[\left(\psi\gamma^{jk}\rho\right)\left(\rho\gamma^{jk}\Pi\right) - (j,k,\rightleftharpoons j',k')\right]\Bigg\} + \mathcal{O}(R^{-4}) \qquad (5.21)\\
\rho^\dagger &= \frac{1}{R^2}\Bigg\{\frac{i}{4}p_-y^2\psi + \frac{i}{8p_-}\left[(p_A^2) + (x'^A)^2\right]\psi + \frac{1}{4p_-^2}\left(p_A x'^A\right)\rho - \frac{1}{4}\left(\rho\Pi\psi\right)\psi \\
&\quad -\frac{1}{8p_-^2}\left(\psi\rho' + \rho\psi'\right)\rho - \frac{1}{8}\left(\psi\psi' - \frac{1}{p_-^2}\rho\rho'\right)\psi\Bigg\} + \mathcal{O}(R^{-4})\ . \qquad (5.22)
\end{aligned}
$$

Denoting the $\mathcal{O}(1/R^2)$ corrections to ρ in (5.21) by Φ, the pp-wave Hamiltonian can be expressed in terms of canonical variables as

$$
\begin{aligned}
\mathcal{H}_{\rm pp} &= -p_-\psi^\dagger\Pi\psi + \frac{i}{2}\psi\psi' + \frac{i}{2}\psi^\dagger\psi'^\dagger \\
&= i\rho\Pi\psi + \frac{i}{2}\psi\psi' - \frac{i}{2p_-^2}\rho\rho' + \frac{1}{R^2}\left\{\frac{i}{2p_-^2}\rho\Phi' + \frac{i}{2p_-^2}\Phi\rho' - i\Phi\Pi\psi\right\}\ . \qquad (5.23)
\end{aligned}
$$

The $\mathcal{O}(1/R^2)$ correction to the Hamiltonian can also be expressed in terms of canonical variables. The overall canonical Hamiltonian can conveniently be broken into its BMN limit ($\mathcal{H}_{\rm pp}$), pure bosonic ($\mathcal{H}_{\rm BB}$), pure fermionic ($\mathcal{H}_{\rm FF}$) and boson-fermion ($\mathcal{H}_{\rm BF}$) interacting subsectors:

$$
\mathcal{H}_{\rm pp} = \frac{p_-}{2}(x^A)^2 + \frac{1}{2p_-}\left[(p_A)^2 + (x'^A)^2\right] + i\rho\Pi\psi + \frac{i}{2}\psi\psi' - \frac{i}{2p_-^2}\rho\rho' \qquad (5.24)
$$

$$
\begin{aligned}
\mathcal{H}_{\rm BB} &= \frac{1}{R^2}\Bigg\{\frac{1}{4p_-}\left[-y^2\left(p_z^2 + z'^2 + 2y'^2\right) + z^2\left(p_y^2 + y'^2 + 2z'^2\right)\right] + \frac{p_-}{8}\left[(x^A)^2\right]^2 \\
&\quad -\frac{1}{8p_-^3}\left\{\left[(p_A)^2\right]^2 + 2(p_A)^2(x'^A)^2 + \left[(x'^A)^2\right]^2\right\} + \frac{1}{2p_-^3}\left(x'^A p_A\right)^2\Bigg\} \qquad (5.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{\rm FF} \;=\; & -\frac{1}{4R^2}\Bigg\{\frac{1}{p_-}\left(\rho\Pi\psi\right)^2+\frac{1}{p_-^3}\left(\rho\Pi\psi\right)\rho\rho'+\frac{1}{2p_-^3}\left(\psi\psi'-\frac{1}{p_-^2}\rho\rho'\right)\rho\rho' \\
& +\frac{1}{2p_-}\left(\psi\psi'-\frac{1}{p_-^2}\rho\rho'\right)(\rho\Pi\psi)+\frac{1}{2p_-^3}\left(\psi\rho'+\rho\psi'\right)\rho'\psi \\
& +\frac{1}{12p_-^3}\left(\psi\gamma^{jk}\rho\right)\left(\rho\gamma^{jk}\Pi\rho'\right) \\
& -\frac{1}{48p_-}\left(\psi\gamma^{jk}\psi-\frac{1}{p_-^2}\rho\gamma^{jk}\rho\right)\left(\rho'\gamma^{jk}\Pi\psi-\rho\gamma^{jk}\Pi\psi'\right)-(j,k\rightleftharpoons j',k')\Bigg\} \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{\rm BF} \;=\; & \frac{1}{R^2}\Bigg\{\frac{i}{4}z^2\psi\psi'-\frac{i}{8p_-^2}\left[(p_A)^2+(x'^A)^2\right]\psi\psi'+\frac{i}{4p_-^4}\left[(p_A)^2+(x'^A)^2+p_-^2(y^2-z^2)\right]\rho\rho' \\
& -\frac{i}{2p_-^2}\left(p_k^2+y'^2-p_-^2z^2-\frac{1}{4}(p_A)^2-\frac{1}{4}(x'^A)^2-\frac{p_-^2}{2}y^2\right)\rho\Pi\psi \\
& +\frac{i}{4}(z_j'z_k)\left(\psi\gamma^{jk}\psi-\frac{1}{p_-^2}\rho\gamma^{jk}\rho\right)-\frac{i}{4}(y_{j'}'y_{k'})\left(\psi\gamma^{j'k'}\psi-\frac{1}{p_-^2}\rho\gamma^{j'k'}\rho\right) \\
& -\frac{i}{8}(z_k'y_{k'}+z_ky_{k'}')\left(\psi\gamma^{kk'}\psi-\frac{1}{p_-^2}\rho\gamma^{kk'}\rho\right)+\frac{1}{4p_-}(p_ky_{k'}+z_kp_{k'})\psi\gamma^{kk'}\rho \\
& +\frac{1}{4p_-}(p_jz_k')\left(\psi\gamma^{jk}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{jk}\Pi\rho\right)-\frac{1}{4p_-}(p_{j'}y_{k'}')\left(\psi\gamma^{j'k'}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{j'k'}\Pi\rho\right) \\
& -\frac{1}{4p_-}(p_ky_{k'}'+z_k'p_{k'})\left(\psi\gamma^{kk'}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{kk'}\Pi\rho\right)-\frac{i}{2p_-^2}(p_kp_{k'}-z_k'y_{k'}')\psi\gamma^{kk'}\Pi\rho \\
& -\frac{1}{4p_-^3}(p_Ax'^A)(\rho\psi'+2\psi\rho')\Bigg\}\,. \qquad (5.27)
\end{aligned}
$$

This Hamiltonian has one problem which we must resolve before attempting to extract its detailed consequences. At the end of section 2, we argued that when the theory is restricted to the subspace of string zero-modes (i.e. excitations of the string that are independent of the worldsheet coordinate σ), curvature corrections to the leading pp-wave Hamiltonian should vanish. The only terms in the Hamiltonian that survive in this limit are those with no worldsheet spatial derivatives. Although $\mathcal{H}_{\rm BB}$ has no such terms, the fermionic pieces of the Hamiltonian do. For example, $\mathcal{H}_{\rm FF}$ contains a term $R^{-2}(\rho\Pi\psi)^2$ which would appear to modify the zero-mode spectrum at $\mathcal{O}(1/R^2)$, contrary to expectation. In the end, we found that this problem can be traced to the presence of second-class constraints involving $\dot{\psi}^\dagger$. As it turns out, the constrained quantization procedure needed to handle second-class constraints has the effect, among many others, of resolving the zero-mode paradox just outlined. To see this, we must work out the appropriate constrained quantization procedure.

The set of constraints that define canonical momenta are known as primary constraints, and take the generic form $\chi = 0$. Primary constraints can be categorized as either first or second class. Second-class constraints arise when canonical momenta do not have vanishing Poisson brackets with the primary constraints themselves: $\{\rho_\psi, \chi_\psi\} \neq 0$, $\{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\} \neq$

0. (First-class constraints are characterized by the more typical condition $\left\{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\right\} = \{\rho_\psi, \chi_\psi\} = 0$.) To the order of interest, the primary constraint equations are

$$\begin{aligned}
\chi^1_\alpha \;=0= \;& \rho_\alpha - ip_-\psi^\dagger_\alpha \\
& -\frac{ip_-}{8R^2}\left[2y^2 + \frac{1}{p_-^2}\left[(p_A)^2 + (x'^A)^2\right] - 2(\psi^\dagger\Pi\psi) + \frac{i}{p_-}(\psi\psi' + \psi^\dagger\psi'^\dagger)\right]\psi^\dagger_\alpha \\
& -\frac{i}{4p_-R^2}\left[(p_A x'^A) - \frac{ip_-}{2}(\psi\psi'^\dagger + \psi^\dagger\psi')\right]\psi_\alpha + \frac{ip_-}{48R^2}(\psi\gamma^{jk}\psi^\dagger)(\psi^\dagger\gamma^{jk}\Pi)_\alpha \quad (5.28)\\
\chi^2_\alpha \;=0= \;& \rho^\dagger_\alpha - \frac{ip_-}{4R^2}\left[y^2 + \frac{1}{2p_-^2}\left[(p_A)^2 + (x'^A)^2\right] - (\psi^\dagger\Pi\psi) + \frac{i}{2p_-}(\psi\psi' + \psi^\dagger\psi'^\dagger)\right]\psi_\alpha \\
& -\frac{i}{4p_-R^2}\left[(p_A x'^A) - \frac{ip_-}{2}(\psi\psi'^\dagger + \psi^\dagger\psi')\right]\psi^\dagger_\alpha \,. \quad (5.29)
\end{aligned}$$

It is clear that these constraints are second-class. In the presence of second-class constraints, consistent quantization requires that the quantum anticommutator of two fermionic fields be identified with their Dirac bracket (which depends on the Poisson bracket algebra of the constraints) rather than with their classical Poisson bracket. The Dirac bracket is given in terms of Poisson brackets by (see, for example, [16])

$$\{A, B\}_{\mathrm{D}} = \{A, B\}_{\mathrm{P}} - \{A, \chi_N\}_{\mathrm{P}} \left(C^{-1}\right)^{NM} \{\chi_M, B\}_{\mathrm{P}} \,, \tag{5.30}$$

where

$$C_{NM} \equiv \{\chi_N, \chi_M\}_{\mathrm{P}} \,. \tag{5.31}$$

The indices N and M denote both the spinor index α and the constraint label $a = 1, 2$. For Grassmanian fields A and B, the Poisson bracket is defined by

$$\{A, B\}_{\mathrm{P}} = -\left(\frac{\partial A}{\partial\psi^\alpha}\frac{\partial B}{\partial\rho_\alpha} + \frac{\partial B}{\partial\psi^\alpha}\frac{\partial A}{\partial\rho_\alpha}\right) - \left(\frac{\partial A}{\partial\psi^{\dagger\alpha}}\frac{\partial B}{\partial\rho^\dagger_\alpha} + \frac{\partial B}{\partial\psi^{\dagger\alpha}}\frac{\partial A}{\partial\rho^\dagger_\alpha}\right) \,. \tag{5.32}$$

As an example, the Dirac bracket $\{\rho_\alpha, \rho_\beta\}_{\mathrm{D}}$ is readily computed (to the order of interest) by noting that the partial integration in (5.16) introduces an asymmetry between ψ and $\psi^\dagger$ into the system. Since $\{\rho_\alpha, \rho_\beta\}_{\mathrm{D}}$ contains

$$\{\rho_\alpha, \chi_{a\gamma}\} = \mathcal{O}(R^{-2}) \qquad \{\chi_{b\eta}, \rho_\beta\} = \mathcal{O}(R^{-2}) \,, \tag{5.33}$$

an immediate consequence of this asymmetry is that $\{\rho_\alpha, \rho_\beta\}_{\mathrm{D}}$ vanishes to $\mathcal{O}(1/R^4)$. To compute $\{\rho_\alpha, \psi_\beta\}_{\mathrm{D}}$, we note that

$$\begin{aligned}
\{\rho_\alpha, \chi_{(2\gamma)}\}_{\mathrm{P}} &= -\delta_{\alpha\rho}\frac{\partial\chi_{(2\gamma)}}{\partial\psi_\rho} \\
&= \mathcal{O}(R^{-2}) \,, \qquad (5.34)
\end{aligned}$$

and, to leading order,

$$(C^{-1})^{(2\gamma)(1\eta)} = -\frac{i}{p_-}\delta_{\gamma\eta} + \mathcal{O}(R^{-2}) \ , \tag{5.35}$$

such that

$$\{\rho_\alpha, \psi_\beta\}_{\rm D} \ = \ -\delta_{\alpha\beta} - \frac{i}{p_-}\{\rho_\alpha, \chi_{(2\beta)}\}_{\rm P} \ . \tag{5.36}$$

Similar manipulations are required for $\{\psi_\alpha, \psi_\beta\}_{\rm D}$, which does exhibit $\mathcal{O}(1/R^2)$ corrections. The second-class constraints on the fermionic sector of the system are removed by enforcing

$$\begin{aligned}
\{\rho_\alpha(\sigma), \psi_\beta(\sigma')\}_{\rm D} \ = \ & -\delta_{\alpha\beta}\delta(\sigma-\sigma') + \frac{1}{4R^2}\delta(\sigma-\sigma')\bigg\{\frac{-i}{p_-}(\rho\Pi)_\alpha\psi_\beta + \frac{i}{p_-}(\rho\Pi\psi)\delta_{\alpha\beta} \\
& + \frac{i}{2p_-}\left[\left(\psi\psi'\delta_{\alpha\beta} - \frac{1}{p_-}{}^2\rho\rho'\delta_{\alpha\beta}\right) + \psi'_\alpha\psi_\beta + \frac{1}{p_-^2}\rho'_\alpha\rho_\beta\right] \\
& + \frac{1}{2p_-^2}\left[(p_A)^2 + (x'^A)^2\right]\delta_{\alpha\beta} + y^2\delta_{\alpha\beta}\bigg\} \\
& - \frac{i}{8p_-R^2}\left(\psi_\alpha\psi_\beta + \frac{1}{p_-^2}\rho_\alpha\rho_\beta\right)\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma') + \mathcal{O}(R^{-4})
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
\{\psi_\alpha(\sigma), \psi_\beta(\sigma')\}_{\rm D} \ = \ & \frac{i}{4p_-R^2}\delta(\sigma-\sigma')\bigg\{(\psi\Pi)_{(\alpha}\psi_{\beta)} - \frac{1}{p_-^2}(p_A x'^A)\delta_{(\alpha\beta)} \\
& + \frac{1}{2p_-^2}\left[\psi'_{(\alpha}\rho_{\beta)} - \rho'_{(\alpha}\psi_{\beta)} + (\psi\rho' + \rho\psi')\delta_{(\alpha\beta)}\right]\bigg\} \\
& + \frac{i}{8p_-^3R^2}\left(\rho_{(\alpha}\psi_{\beta)} - \psi_{(\alpha}\rho_{\beta)}\right)\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma') + \mathcal{O}(R^{-4})
\end{aligned} \tag{5.38}$$

$$\{\rho_\alpha(\sigma), \rho_\beta(\sigma')\}_{\rm D} \ = \ \mathcal{O}(R^{-4}) \ . \tag{5.39}$$

Identifying these Dirac brackets with the quantum anticommutators of the fermionic fields in the theory naturally leads to additional $\mathcal{O}(1/R^2)$ corrections to the energy spectrum. One way to implement these corrections is to retain the Fourier expansion of ψ and $\psi^\dagger$ given in (5.7,5.8) while transforming the fermionic creation and annihilation operators

$$b_n^\alpha \to c_n^\alpha \qquad\qquad b_n^{\dagger\alpha} \to c_n^{\dagger\alpha} \ , \tag{5.40}$$

such that $\{\rho(c, c^\dagger), \psi(c, c^\dagger)\}_{\rm P}$, for example, satisfies (5.37). This approach amounts to finding $\mathcal{O}(1/R^2)$ corrections to $\{c_n^\alpha, c_m^{\dagger\beta}\}$ that allow the usual anticommutators to be identified with the above Dirac brackets (5.37-5.39). In practice, extracting these solutions from (5.37-5.39) can be circumvented by invoking a non-linear field redefinition $\psi \to \tilde\psi, \ \rho \to \tilde\rho$, such that

$$\{\rho(c, c^\dagger), \psi(c, c^\dagger)\}_{\rm P} = \{\tilde\rho(b, b^\dagger), \tilde\psi(b, b^\dagger)\}_{\rm P} \ . \tag{5.41}$$

Both representations satisfy (5.37), and the operators $b_n^\alpha, b_m^{\dagger\beta}$ are understood to obey the usual relations:

$$\{b_n^\alpha, b_m^{\dagger\beta}\} = \delta^{\alpha\beta}\delta_{nm} \ . \tag{5.42}$$

In general, the non-linear field redefinition $\tilde{\psi}(b,b^\dagger) = \psi(b,b^\dagger) + \ldots$ contains corrections that are cubic in the fields $\rho(b,b^\dagger)$, $\psi(b,b^\dagger)$, $x^A(a,a^\dagger)$ and $p_A(a,a^\dagger)$. Such correction terms can be written down by inspection, with matrix-valued coefficients to be solved for by comparing $\{\tilde\rho(b,b^\dagger), \tilde\psi(b,b^\dagger)\}_{\rm P}$ and $\{\tilde\psi(b,b^\dagger), \tilde\psi(b,b^\dagger)\}_{\rm P}$ with (5.37,5.38). A straightforward computation yields

$$\begin{aligned}
\rho_\alpha \to \tilde\rho_\alpha &= \rho_\alpha && (5.43)\\
\psi_\beta \to \tilde\psi_\beta &= \psi_\beta + \frac{i}{8p_-R^2}\Big\{(\psi'\psi)\psi_\beta - 2(\rho\Pi\psi)\psi_\beta - \frac{1}{p_-^2}(\rho'\rho)\psi_\beta + \frac{2}{p_-^2}(p_A x'^A)\rho_\beta \\
&\quad + \frac{1}{p_-^2}\left[(\rho'\psi)\rho_\beta - (\rho\psi')\rho_\beta\right] + 2ip_-\left[y^2\psi_\beta + \frac{1}{2p_-^2}\left((p_A)^2 + (x'^A)^2\right)\psi_\beta\right]\Big\} \ . && (5.44)
\end{aligned}$$

This approach to enforcing the modified Dirac bracket structure amounts to adding $\mathcal{O}(1/R^2)$ correction terms to the Hamiltonian while keeping the standard commutation relations. It is much more convenient for calculating matrix elements than the alternative approach of adding $\mathcal{O}(1/R^2)$ operator corrections to the fermi field anticommutators $\{b, b^\dagger\}$.

By invoking the redefinitions in (5.43,5.44), the pieces of the interaction Hamiltonian that involve fermions take the final forms

$$\begin{aligned}
\mathcal{H}_{\rm FF} &= -\frac{1}{4p_-^3R^2}\Big\{p_-^2\left[(\psi'\psi) + \frac{1}{p_-^2}(\rho\rho')\right](\rho\Pi\psi) - \frac{p_-^2}{2}(\psi'\psi)^2 - \frac{1}{2p_-^2}(\rho'\rho)^2 + (\psi'\psi)(\rho'\rho) \\
&\quad + (\rho\psi')(\rho'\psi) - \frac{1}{2}\left[(\psi\rho')(\psi\rho') + (\psi'\rho)^2\right] + \frac{1}{12}(\psi\gamma^{jk}\rho)(\rho\gamma^{jk}\Pi\rho') \\
&\quad - \frac{p_-^2}{48}\left(\psi\gamma^{jk}\psi - \frac{1}{p_-^2}\rho\gamma^{jk}\rho\right)\left(\rho'\gamma^{jk}\Pi\psi - \rho\gamma^{jk}\Pi\psi'\right) - (j,k \rightleftharpoons j',k')\Big\} \ ,
\end{aligned} \tag{5.45}$$

$$
\begin{aligned}
\mathcal{H}_{\rm BF} \;=\; & \frac{1}{R^2}\Bigg\{-\frac{i}{4p_-^2}\left[(p_A)^2+(x'^A)^2+p_-^2(y^2-z^2)\right]\left(\psi\psi'-\frac{1}{p_-^2}\rho\rho'\right) \\
& -\frac{1}{2p_-^3}(p_Ax'^A)(\rho\psi'+\psi\rho')-\frac{i}{2p_-^2}\left(p_k^2+y'^2-p_-^2z^2\right)\rho\Pi\psi \\
& +\frac{i}{4}(z'_jz_k)\left(\psi\gamma^{jk}\psi-\frac{1}{p_-^2}\rho\gamma^{jk}\rho\right)-\frac{i}{4}(y'_{j'}y_{k'})\left(\psi\gamma^{j'k'}\psi-\frac{1}{p_-^2}\rho\gamma^{j'k'}\rho\right) \\
& -\frac{i}{8}(z'_ky_{k'}+z_ky'_{k'})\left(\psi\gamma^{kk'}\psi-\frac{1}{p_-^2}\rho\gamma^{kk'}\rho\right)+\frac{1}{4p_-}(p_ky_{k'}+z_kp_{k'})\psi\gamma^{kk'}\rho \\
& +\frac{1}{4p_-}(p_jz'_k)\left(\psi\gamma^{jk}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{jk}\Pi\rho\right)-\frac{1}{4p_-}(p_{j'}y'_{k'})\left(\psi\gamma^{j'k'}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{j'k'}\Pi\rho\right) \\
& -\frac{1}{4p_-}(p_ky'_{k'}+z'_kp_{k'})\left(\psi\gamma^{kk'}\Pi\psi+\frac{1}{p_-^2}\rho\gamma^{kk'}\Pi\rho\right)-\frac{i}{2p_-^2}(p_kp_{k'}-z'_ky'_{k'})\psi\gamma^{kk'}\Pi\rho\Bigg\}\ .
\end{aligned}
\tag{5.46}
$$

The full Hamiltonian is the sum of these two terms plus the bosonic interaction term $\mathcal{H}_{\rm BB}$ (5.25) and the free Hamiltonian $\mathcal{H}_{\rm pp}$ (5.24). This system is quantized by imposing the standard (anti)commutator algebra for x^A, ψ and their conjugate variables p^A, ρ. This will be done by expanding the field variables in creation and annihilation operators in a standard way.

Returning to the phenomenon that led us to explore second-class constraints in the first place, note that (5.45) manifestly vanishes on the subspace of string zero-modes because all terms have at least one worldsheet spatial derivative. The bose-fermi mixing Hamiltonian (5.46) still has terms which can lead to curvature corrections to the string zero-mode energies, but their net effect vanishes by virtue of non-trivial cancellations between terms that split $SO(4)\times SO(4)$ indices and terms that span the entire $SO(8)$. How this comes about will be seen when we actually compute matrix elements of this Hamiltonian.

6 Perturbative analysis of the string energy spectrum

To compute the energy spectrum correct to first order in $\mathcal{O}(R^{-1})$, we will do degenerate first-order perturbation theory on the Fock space of eigenstates of the free Hamiltonian $\mathcal{H}_{\rm pp}$. The degenerate subspaces of the BMN theory are spanned by fixed numbers of creation operators with specified mode indices (subject to the constraint that the mode indices sum up to zero) acting on the ground state $|J\rangle$, where $J = p_-R^2$ is the angular momentum (assumed large) of the string center of mass in its motion around the equator of the S^5. In this paper we restrict attention to "two-impurity states" generated by pairs of creation operators of equal and opposite mode number. For each positive mode number n, the 16 bosonic and fermionic creation operators can be combined in pairs to form the following 256 degenerate "two-impurity" states:

$$
a_n^{A\dagger}a_{-n}^{B\dagger}|J\rangle \qquad b_n^{\alpha\dagger}b_{-n}^{\beta\dagger}|J\rangle \qquad a_n^{A\dagger}b_{-n}^{\alpha\dagger}|J\rangle \qquad a_{-n}^{A\dagger}b_n^{\alpha\dagger}|J\rangle\ . \tag{6.1}
$$

The creation operators are classified under the residual $SO(4) \times SO(4)$ symmetry to which the isometry group of the $AdS_5 \times S^5$ target space is broken by the lightcone gauge quantization procedure. The bosonic creation operators $a_n^{A\dagger}$ decompose as $(\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})$, or, in the $SU(2)^2 \times SU(2)^2$ notation introduced in [4], as $(\mathbf{2}, \mathbf{2}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$. Analogously, the fermionic operators $b_n^{\alpha\dagger}$ decompose as $(\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ under the covering group. It is useful to note that the two fermion irreps are eigenvectors, with opposite eigenvalue, of the Π operator introduced in (3.6). To find the perturbed energy spectrum, we must compute explicit matrix elements of $\mathcal{H}_{\rm int}$ in this basis and then diagonalize the resulting 256×256 matrix. We will compare the perturbed energy eigenvalues with general expectations from $PSU(2,2|4)$ as well as with the large $\mathcal{R}$-charge limit of the anomalous dimensions of gauge theory operators with two $\mathcal{R}$-charge defects. Higher-impurity string states can be treated in the same way, but we defer such questions to a separate paper [17]. Our purpose here is primarily to check that our methods (choice of action, light-cone gauge reduction, quantization rules, etc.) are consistent and correct. Due to the algebraic complexity met with at each step, this check is far from trivial. Once reassured on these fundamental points, we can go on to examine a wider range of physically interesting issues.

The first step in carrying out this program is to expand $\mathcal{H}_{\rm int}$ in creation and annihilation operators using (5.2,5.7) for x^A, ψ and the related expansions for p^A, ρ. As an example, we quote the result for $H_{\rm BB}$ (keeping only terms with two creation and two annihilation operators):

$$\begin{aligned}
\mathcal{H}_{\rm BB} \;=\; & -\frac{1}{32 p_- R^2} \sum \frac{\delta(n+m+l+p)}{\xi} \times \\
& \Big\{ 2 \Big[\xi^2 - (1 - k_l k_p k_n k_m) + \omega_n \omega_m k_l k_p + \omega_l \omega_p k_n k_m + 2\omega_n \omega_l k_m k_p \\
& + 2\omega_m \omega_p k_n k_l \Big] a_{-n}^{\dagger A} a_{-m}^{\dagger A} a_l^B a_p^B + 4 \Big[\xi^2 - (1 - k_l k_p k_n k_m) - 2\omega_n \omega_m k_l k_p + \omega_l \omega_m k_n k_p \\
& - \omega_n \omega_l k_m k_p - \omega_m \omega_p k_n k_l + \omega_n \omega_p k_m k_l \Big] a_{-n}^{\dagger A} a_{-l}^{\dagger B} a_m^A a_p^B + 2 \Big[8 k_l k_p a_{-n}^{\dagger i} a_{-l}^{\dagger j} a_m^i a_p^j \\
& + 2(k_l k_p + k_n k_m) a_{-n}^{\dagger i} a_{-m}^{\dagger i} a_l^j a_p^j + (\omega_l \omega_p + k_l k_p - \omega_n \omega_m - k_n k_m) a_{-n}^{\dagger i} a_{-m}^{\dagger i} a_l^{j'} a_p^{j'} \\
& - 4(\omega_l \omega_p - k_l k_p) a_{-n}^{\dagger i} a_{-l}^{\dagger j'} a_m^i a_p^{j'} - (i, j \rightleftharpoons i', j') \Big] \Big\} \,, \qquad (6.2)
\end{aligned}$$

with $\xi \equiv \sqrt{\omega_n \omega_m \omega_l \omega_p}$. The expansion of the interaction terms involving fermi fields are too complicated to be worth writing down explicitly at this stage. Schematically, we organize the two-impurity matrix elements of the perturbing Hamiltonian as shown in Table 1.

To organize the perturbation theory, it is helpful to express everything in terms of two parameters: J and λ'. In the duality between Type IIB superstring theory on $AdS_5 \times S^5$

$(\mathcal{H})_{int}$	$a_n^{A\dagger}a_{-n}^{B\dagger}\|J\rangle$	$b_n^{\alpha\dagger}b_{-n}^{\beta\dagger}\|J\rangle$	$a_n^{A\dagger}b_{-n}^{\alpha\dagger}\|J\rangle$	$a_{-n}^{A\dagger}b_n^{\alpha\dagger}\|J\rangle$
$\langle J\| a_n^A a_{-n}^B$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BF}$	0	0
$\langle J\| b_n^\alpha b_{-n}^\beta$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm FF}$	0	0
$\langle J\| a_n^A b_{-n}^\alpha$	0	0	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$
$\langle J\| a_{-n}^A b_n^\alpha$	0	0	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$

Table 1: Structure of the matrix of first-order energy perturbations in the space of two-impurity string states

and $\mathcal{N}=4$ $SU(N_c)$ super-Yang-Mills theory in four dimensions, we identify

$$\begin{array}{rcl} \mathcal{N}=4\ \text{SYM} & & AdS_5\times S^5 \\ SU(N_c) & \rightleftharpoons & \int_{S^5} F_5 = N_c \\ g_{\rm YM}^2 N_c & \rightleftharpoons & R^4 \\ g_{\rm YM}^2 & \rightleftharpoons & g_s . \end{array} \tag{6.3}$$

In the pp-wave limit, however, the AdS/CFT dictionary reads

$$\begin{array}{rcl} \mathcal{R} & \rightleftharpoons & p_- R^2 = J \\ \dfrac{\mathcal{R}^2}{N_c} & \rightleftharpoons & g_s p_-^2 = g_2 \\ \mathcal{R}\to\infty & \rightleftharpoons & p_- R^2, N_c\to\infty\ . \end{array} \tag{6.4}$$

The modified 't Hooft coupling

$$\lambda' = \frac{g_{\rm YM}^2 N_c}{\mathcal{R}^2} \rightleftharpoons \frac{1}{p_-^2} \tag{6.5}$$

is kept fixed in the $\mathcal{R}, N_c\to\infty$ limit. (We have kept $\alpha'=\mu=1$.) Since the gauge theory is perturbative in $\lambda = g_{YM}^2 N_c$, and p_-^2 on the string side is mapped to $\mathcal{R}^2/(g_{YM}^2 N_c)$, we will expand string energies ω_q in powers of $1/p_-$, keeping terms up to some low order to correspond with the loop expansion in the gauge theory. This type of dictionary would be incorrect in the original coordinate system characterized by the light-cone coordinates $t = x^+ - (x^-/2R^2)$ and $\phi = x^+ + (x^-/2R^2)$ given in (2.4). In this case, one would calculate corrections to $\mathcal{R}\rightleftharpoons p_- R^2$ appearing in the perturbing Hamiltonian (which amount to operator-valued corrections to p_-).

6.1 Evaluating Fock space matrix elements of $\mathcal{H}_{\rm BB}$

We now proceed to the construction of the perturbing Hamiltonian matrix on the space of degenerate two-impurity states. To convey a sense of what is involved, we display the matrix

elements of $\mathcal{H}_{\rm BB}$ (5.25) between the bosonic two-impurity Fock space states:

$$\begin{aligned}
\Big\langle J \Big| a_n^A a_{-n}^B \left(\mathcal{H}_{\rm BB}\right) a_{-n}^{C\dagger} a_n^{D\dagger} \Big| J \Big\rangle \;&=\; \left(N_{\rm BB}(n^2\lambda') - 2n^2\lambda'\right) \frac{\delta^{AD}\delta^{BC}}{J} \\
&+ \quad \frac{n^2\lambda'}{J(1+n^2\lambda')} \left[\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right] \\
&- \quad \frac{n^2\lambda'}{J(1+n^2\lambda')} \left[\delta^{a'b'}\delta^{c'd'} + \delta^{a'd'}\delta^{b'c'} - \delta^{a'c'}\delta^{b'd'}\right] \\
\approx (n_{\rm BB} - 2)\, \frac{n^2\lambda'}{J}\delta^{AD}\delta^{BC} &+ \frac{n^2\lambda'}{J} \left[\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right] \\
-\frac{n^2\lambda'}{J} \quad &\left[\delta^{a'b'}\delta^{c'd'} + \delta^{a'd'}\delta^{b'c'} - \delta^{a'c'}\delta^{b'd'}\right] + \mathcal{O}(\lambda'^2)\ , \qquad (6.6)
\end{aligned}$$

where lower-case $SO(4)$ indices $a,b,c,d \in 1,\ldots,4$ indicate that A,B,C,D are chosen from the first $SO(4)$, and $a',b',c',d' \in 5,\ldots,8$ indicate the second $SO(4)$ $(A,B,C,D \in 5,\ldots,8)$. We have also displayed the further expansion of these $\mathcal{O}(1/J)$ matrix elements in powers of λ' (using the basic BMN-limit energy eigenvalue condition $\omega_n/p_- = \sqrt{1+\lambda' n^2}$). This is to facilitate eventual contact with perturbative gauge theory via AdS/CFT duality. Note that $\mathcal{H}_{\rm BB}$ does not mix states built out of oscillators from different $SO(4)$ subgroups. There is a parallel no-mixing phenomenon in the gauge theory: two-impurity bosonic operators carrying spacetime vector indices do not mix with spacetime scalar bosonic operators carrying $\mathcal{R}$-charge vector indices.

Due to operator ordering ambiguities, two-impurity matrix elements of $\mathcal{H}_{\rm BB}$ can differ by contributions proportional to $\delta^{AD}\delta^{BC}$, depending on the particular prescription chosen [4]. $N_{\rm BB}(n^2\lambda')$ is an arbitrary function of $n^2\lambda'$ which is included to account for such ambiguities (we will shortly succeed in fixing it). To match the dual gauge theory physics, it is best to expand $N_{\rm BB}$ as a power series in λ'. The zeroth-order term must vanish if the energy correction is to be perturbative in the gauge coupling. The next term in the expansion contributes one arbitrary constant (the $n_{\rm BB}$ term) and each higher term in the λ' expansion in principle contributes one additional arbitrary constant to this sector of the Hamiltonian. Simple general considerations will fix them all.

6.2 Evaluating Fock space matrix elements of $\mathcal{H}_{\rm FF}$

The calculation of the two-impurity matrix elements of the parts of $\mathcal{H}_{\rm int}$ that involve fermionic fields is rather involved and we found it necessary to employ symbolic manipulation programs to keep track of the many different terms. The end results are fairly concise, however. For

$\mathcal{H}_{FF}$ we find

$$\begin{aligned}
\left\langle J \middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm FF}\right) b_{-n}^{\gamma\dagger} b_n^{\delta\dagger} \middle| J \right\rangle &= \left(N_{\rm FF}(n^2\lambda') - 2n^2\lambda'\right) \frac{\delta^{\alpha\delta}\delta^{\beta\gamma}}{J} \\
&+ \frac{n^2\lambda'}{24J(1+n^2\lambda')} \left[(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}\right] \\
&- \frac{n^2\lambda'}{24J(1+n^2\lambda')} \left[(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma} + (\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta} - (\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\right] \\
&\approx (n_{\rm FF} - 2)\, \frac{n^2\lambda'}{J} \delta^{\alpha\delta}\delta^{\beta\gamma} + \frac{n^2\lambda'}{24J} \left[(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}\right] \\
&- \frac{n^2\lambda'}{24J} \left[(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma} + (\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta} - (\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\right] + \mathcal{O}(\lambda'^2)\ .
\end{aligned} \tag{6.7}$$

This sector has its own normal-ordering function $N_{\rm FF}$, with properties similar those of $N_{\rm BB}$ described above. The index structure of the fermionic matrix elements is similar to that of its bosonic counterpart (6.6).

We will now introduce some useful projection operators that will help us understand the selection rules implicit in the index structure of (6.7). The original 16-component spinors ψ were reduced to 8 components by the Weyl condition $\bar\gamma^9\psi = \psi$. The remaining 8 components are further divided into spinors $\tilde\psi$ and $\hat\psi$ which are even or odd under the action of Π:

$$\begin{aligned}
\Pi\tilde\psi &= -\tilde\psi \qquad\qquad \Pi\tilde b^{\dagger\alpha} = -\tilde b^{\dagger\alpha} \\
\Pi\hat\psi &= \hat\psi \qquad\qquad \Pi\hat b^{\dagger\alpha} = \hat b^{\dagger\alpha}\ .
\end{aligned} \tag{6.8}$$

The spinors $\hat\psi$ transform in the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ of $SO(4)\times SO(4)$, while $\tilde\psi$ transform in the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$. This correlation between Π-parity and $SO(4)\times SO(4)$ representation will be very helpful for analyzing complicated fermionic matrix elements.

We denote the $SU(2)$ generators of the active factors of the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ irrep as Σ^+ and Ω^+, where the Σ act on the SO(4) descended from the AdS_5, and the Ω act on the $SO(4)$ coming from the S^5. The $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ generators are similarly labeled by Σ^- and Ω^-. Each set of spinors is annihilated by its counterpart set of $SU(2)$ generators:

$$\begin{aligned}
\Sigma^+\hat b^{\dagger\alpha} = \Omega^+\hat b^{\dagger\alpha} &= 0 \\
\Sigma^-\tilde b^{\dagger\alpha} = \Omega^-\tilde b^{\dagger\alpha} &= 0\ .
\end{aligned} \tag{6.9}$$

In terms of the projection operators

$$\Pi_+ = \frac{1}{2}(1+\Pi) \qquad \Pi_- = \frac{1}{2}(1-\Pi)\ , \tag{6.10}$$

which select the disjoint $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ irreps, respectively, we have

$$\begin{aligned}
\Pi_+\psi &= \hat\psi \qquad\qquad \Pi_+\hat b^\alpha = \hat b^\alpha \\
\Pi_-\psi &= \tilde\psi \qquad\qquad \Pi_-\tilde b^\alpha = \tilde b^\alpha\ .
\end{aligned} \tag{6.11}$$

The $\Pi_\pm$ projections commute with the $SO(4)$ generator matrices $\gamma^{ij}, \gamma^{i'j'}$, a fact which implies certain useful selection rules for the one-loop limit of (6.7). The rules are most succinctly stated using an obvious $\pm$ shorthand to indicate the representation content of states created by multiple fermionic creation operators. In brief, one finds that $++$ states connect only with $++$ and $--$ states connect only with $--$. The only subtle point is the statement that all $++ \to --$ matrix elements of (6.7) must vanish: this is the consequence of a simple cancellation between two terms. This observation will simplify the matrix diagonalization we will eventually carry out.

6.3 Evaluating Fock space matrix elements of $\mathcal{H}_{\rm BF}$

The $\mathcal{H}_{\rm BF}$ sector in the Hamiltonian mediates mixing between spacetime bosons of the two types (pure boson and bi-fermion) as well as between spacetime fermions (which of course contain both bosonic and fermionic oscillator excitations). The 64-dimensional boson mixing matrix

$$\left\langle J \left| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm BF}\right) a_{-n}^{A\dagger} a_n^{B\dagger} \right| J \right\rangle \ ,$$

is an off-diagonal block in the bosonic sector of the perturbation matrix in Table 1. The same methods used earlier in this section to reduce Fock space matrix elements involving fermi fields can be used here to obtain the simple explicit result (we omit the details)

$$\begin{aligned}
\left\langle J \left| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm BF}\right) a_{-n}^{A\dagger} a_n^{B\dagger} \right| J \right\rangle \ &= \ \frac{n^2\lambda'}{2J(1+n^2\lambda')}\Big\{\sqrt{1+n^2\lambda'}\Big[\left(\gamma^{ab'}\right)^{\alpha\beta} - \left(\gamma^{a'b}\right)^{\alpha\beta}\Big] \\
&\quad + n\sqrt{\lambda'}\left[\left(\gamma^{a'b'}\right)^{\alpha\beta} - \left(\gamma^{ab}\right)^{\alpha\beta} + \left(\delta^{ab} - \delta^{a'b'}\right)\delta^{\alpha\beta}\right]\Big\} \\
&\approx \ \frac{n^2\lambda'}{2J}\left[\left(\gamma^{ab'}\right)^{\alpha\beta} - \left(\gamma^{a'b}\right)^{\alpha\beta}\right] + \mathcal{O}(\lambda'^{3/2}) \ . \qquad (6.12)
\end{aligned}$$

The complex conjugate of this matrix element gives the additional off-diagonal component of the upper 128×128 block of spacetime bosons. We note that terms in the $\mathcal{H}_{\rm BF}$ sector split the $SO(8)$ group (manifest in the pp-wave limit) into its $SO(4)$ constituents such that states of the form $a_{-n}^{a'\dagger} a_n^{b'\dagger}\left|J\right\rangle$, for example, which descend strictly from the S^5 subspace, vanish in this subsector. This behavior is reproduced in the gauge theory, wherein two-boson states that are either spacetime scalars or scalars of the $\mathcal{R}$-charge group do not mix with bi-fermionic scalars in either irrep.

The 128-dimensional subsector of spacetime fermions is mixed by matrix elements of the same Hamiltonian taken between fermionic string states of the general form $b_n^{\alpha\dagger} a_{-n}^{A\dagger}\left|J\right\rangle$. Our standard methods yield the following simple results for the two independent types of

spacetime fermion mixing matrix elements:

$$\begin{aligned}
\left\langle J\middle| b_n^{\alpha} a_{-n}^{A}\left(\mathcal{H}_{\rm BF}\right) b_n^{\beta\dagger} a_{-n}^{B\dagger}\middle| J\right\rangle &= N_{\rm BF}(n^2\lambda')\frac{\delta^{AB}\delta^{\alpha\beta}}{J} \\
&+\frac{n^2\lambda'}{2J(1+n^2\lambda')}\left\{\left(\gamma^{ab}\right)^{\alpha\beta}-\left(\gamma^{a'b'}\right)^{\alpha\beta}-(3+4n^2\lambda')\delta^{ab}\delta^{\alpha\beta}-(5+4n^2\lambda')\delta^{a'b'}\delta^{\alpha\beta}\right\} \\
&\approx \frac{n^2\lambda'}{2J}\left\{\left(\gamma^{ab}\right)^{\alpha\beta}-\left(\gamma^{a'b'}\right)^{\alpha\beta}+\left[(2n_{\rm BF}-3)\delta^{ab}+(2n_{\rm BF}-5)\delta^{a'b'}\right]\delta^{\alpha\beta}\right\}+\mathcal{O}(\lambda'^2)\ ,
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
\left\langle J\middle| b_n^{\alpha} a_{-n}^{A}\left(\mathcal{H}_{\rm BF}\right) b_{-n}^{\beta\dagger} a_{n}^{B\dagger}\middle| J\right\rangle &= \frac{n^2\lambda'}{2J\sqrt{1+n^2\lambda'}}\left\{\left(\gamma^{ab}\right)^{\alpha\beta}-\left(\gamma^{a'b'}\right)^{\alpha\beta}\right. \\
&\left.-\frac{n\lambda'^{1/2}}{\sqrt{1+n^2\lambda'}}\left[\left(\gamma^{ab'}\right)^{\alpha\beta}-\left(\gamma^{a'b}\right)^{\alpha\beta}\right]-\delta^{\alpha\beta}\left(\delta^{ab}-\delta^{a'b'}\right)\right\} \\
&\approx \frac{n^2\lambda'}{2J}\left\{\left(\gamma^{ab}\right)^{\alpha\beta}-\left(\gamma^{a'b'}\right)^{\alpha\beta}-\left(\delta^{ab}-\delta^{a'b'}\right)\delta^{\alpha\beta}\right\}+\mathcal{O}(\lambda'^{3/2})\ .
\end{aligned} \tag{6.14}$$

Equation (6.13) involves yet another normal-ordering function. Since these functions have a non-trivial effect on the spectrum, we must give them specific values before we can calculate actual numerical eigenvalues. The key point is that the structure of the perturbing Hamiltonian implies certain relations between all the normal-ordering functions. Because the interaction Hamiltonian is quartic in oscillators, normal-ordering ambiguities give rise to terms quadratic in oscillators, appearing as constant contributions to the diagonal matrix elements. There are normal-ordering contributions from each sector of the theory: $\mathcal{H}_{\rm BB}$ contributes a single term quadratic in bosonic oscillators; $\mathcal{H}_{\rm FF}$ yields a term quadratic in fermionic oscillators; $\mathcal{H}_{\rm BF}$ contributes one term quadratic in bosons and one quadratic in fermions. The bosonic contributions multiply terms of the form $a^\dagger a$, which are collected into the function $N_{\rm BB}(n^2\lambda')$ with one contribution from $\mathcal{H}_{\rm BB}$ and one contribution from $\mathcal{H}_{\rm BF}$. Similarly, $N_{\rm FF}(n^2\lambda')$ collects terms multiplying $b^\dagger b$, receiving one contribution from $\mathcal{H}_{\rm FF}$ and one contribution from $\mathcal{H}_{\rm BF}$. Normal-ordering contributions from both $a^\dagger a$ and $b^\dagger b$ terms are non-vanishing in the spacetime fermion subsector; all possible normal-ordering ambiguities appear in this subspace. The normal-ordering function $N_{\rm BF}(n^2\lambda')$ therefore must satisfy

$$N_{\rm BF}(n^2\lambda') = N_{\rm BB}(n^2\lambda') + N_{\rm FF}(n^2\lambda')\ . \tag{6.15}$$

The normal ordering functions are basically finite renormalizations which must be adjusted so that the spectrum reflects the $PSU(2,2|4)$ global supersymmetry of the classical worldsheet action (a symmetry we want to preserve at the quantum level).

As has been explained elsewhere [18, 4] (and as we shall shortly review), energy levels should be organized into multiplets obtained by acting on a 'highest-weight' level with all possible combinations of the eight $\mathcal{R}$-charge raising supercharges. All the states obtained by acting with a total of L supercharges have the same energy and we will refer to them as states

at level L in the supermultiplet. The levels of a multiplet run from $L = 0$ to $L = 8$. A careful inspection of the way the normal ordering functions contribute to the energies of states in the two-impurity sector shows that states at levels $L = 0, 8$ are shifted by $N_{\rm BB}$ only. Similarly, levels $L = 2, 4, 6$ are shifted by $N_{\rm FF}$ or $N_{\rm BB}$ and one must have $N_{\rm BB} = N_{\rm FF}$ if those levels are to remain internally degenerate. Finally, levels $L = 1, 3, 5, 7$ are shifted by $N_{\rm BF}$ only. By supersymmetry, the level spacing must be uniform throughout the supermultiplet and this is only possible if we also set $N_{\rm BB} = N_{\rm BF}$. But then the constraint $N_{\rm BF} = N_{\rm BB} + N_{\rm FF}$ can only be met by setting $N_{\rm BB} = N_{\rm FF} = N_{\rm BF} = 0$, which then eliminates any normal-ordering ambiguity from the string theory. This is basically an exercise in using global symmetry conditions to fix otherwise undetermined finite renormalizations.

6.4 Diagonalizing the one-loop perturbation matrix

We are now ready to diagonalize the perturbing Hamiltonian and examine whether the resulting energy shifts have the right multiplet structure and whether the actual eigenvalues match gauge theory expectations. To simplify the problem, we will begin by diagonalizing the perturbation matrix expanded to first nontrivial order in both $1/J$ and λ'. Our results should, by duality, match one-loop gauge theory calculations and we will eventually return to the problem of finding the string spectrum correct to higher orders in λ'. From the structure of the results just obtained for the perturbation matrices, we can see that the general structure of the energy eigenvalues of two-impurity states must be

$$E_{\rm int}(n) = 2 + n^2\lambda' \left(1 + \frac{\Lambda}{J} + \mathcal{O}(J^{-2})\right) + \mathcal{O}(\lambda'^2) \ , \tag{6.16}$$

where Λ is dimensionless and the dependence on $1/J$, λ' and mode number n is given by (6.6,6.7). The eigenvalues Λ must meet certain conditions if the requirements of $PSU(2,2|4)$ symmetry are to be met, and we will state those conditions before solving the eigenvalue problem.

Note that the eigenvalues in question are lightcone energies and thus dual to the gauge theory quantity $\Delta = D - J$, the difference between scaling dimension and $\mathcal{R}$-charge. Since conformal invariance is part of the full symmetry group, states are organized into conformal multiplets built on conformal primaries. A supermultiplet will contain several conformal primaries having the same value of Δ and transforming into each other under the supercharges. All 16 supercharges increment the dimension of an operator by 1/2, but only 8 of them (call them Q_α) also increment the $\mathcal{R}$-charge by 1/2, so as to leave Δ unchanged. These 8 supercharges act as 'raising operators' on the conformal primaries of a supermultiplet: starting from a super-primary of lowest $\mathcal{R}$-charge, the other conformal primaries are created by acting on it in all possible ways with the eight Q_α. Primaries obtained by acting with L factors of Q_α on the super-primary are said to be at level L in the supermultiplet (since the Q_α anticommute, the range is $L = 0$ to $L = 8$). The multiplicities of states at the various levels are then determined: for every $L = 0$ primary, there will in general be C^8_L primaries at level L (where C^n_m is the binomial coefficient) and a total of $2^8 = 256$ conformal primaries

summed over all L. If the $L = 0$ conformal primary is not a singlet, the total number of conformal primary states will be a multiple of 256. Since the number of two-impurity string states is exactly 256, we expect the super-primary level to be a singlet (in both spacetime and the residual $SO(4)$ $\mathcal{R}$-symmetry) and therefore necessarily a spacetime boson. This is the translation into string theory language of Beisert's careful analysis of supermultiplets of two-impurity BMN operators in $\mathcal{N} = 4$ super Yang Mills theory [18].

These facts severely restrict the quantity Λ in the general expression (6.16) above. Although the two-impurity states in question have the same J, they in fact belong to different levels L in different supermultiplets. A state of given L is a member of a supermultiplet built on a 'highest-weight' or super-primary state with $\mathcal{R} = J - L/2$. Since all the primaries in a supermultiplet have the same Δ, the joint dependence of eigenvalues on λ, J, L must be of the form $\Delta(\lambda, J - L/2)$. The only way the expansion of (6.16) can be consistent with this is if $\Lambda = L + c$, where c is a pure numerical constant (recall that $\lambda' = \lambda/J^2$). Successive spacetime boson (or successive spacetime fermion) members of a supermultiplet must therefore have eigenvalues separated by exactly 2. We furthermore know that the multiplicity of the level-L eigenvalue must be $C^8_L = 1, 8, 28, ..., 1$ for $L = 0, 1, 2, ..., 8$. The representation content of the different levels under the $SO(3,1)$ spacetime and residual $SO(4)$ $\mathcal{R}$ symmetries can of course also be specified, if desired. Our program, then, is the following: we will first verify that the quantization procedure preserves the $PSU(2,2|4)$ supersymmetry by showing that the eigenvalues Λ satisfy the integer spacing and multiplicity rules just enumerated; in the process we will obtain specific values for Λ which we will then compare with what is known about one-loop gauge theory operator dimensions in order to check the gauge theory duality conjecture. The two issues are logically disconnected: the quantized string theory should make sense on its own terms, whether or not it satisfies the more stringent requirements of duality with four-dimensional gauge theory.

6.5 Details of the one-loop diagonalization procedure.

We now confront the problem of explicitly diagonalizing the first-order perturbation matrix Λ (obtained by expanding the relevant matrix elements to first order in λ'). The matrix block diagonalizes on the spacetime boson and spacetime fermion subspaces, as indicated in Table 1. Within these sub-blocks, there are further block diagonalizations arising from special properties of the one-loop form of the matrix elements of the perturbing Hamiltonian. For example, Fock space states built out of two bosonic creation operators that transform only under the internal $SO(4)$ mix only with themselves, thus providing a 16×16 dimensional diagonal sub-block. Within such sub-blocks, symmetry considerations are often sufficient to completely diagonalize the matrix or at least to reduce it to a low-dimensional diagonalization problem. In short, the problem reduces almost entirely to that of projecting the matrix elements of $\mathcal{H}_{\rm int}$ on subspaces of the two-impurity Fock space defined by various symmetry properties. Determining the $SO(4) \times SO(4)$ symmetry labels of each eigenstate in the diagonalization will furthermore enable us to precisely match string states with gauge theory operators. In this subsection, we record for future reference the detailed arguments

for the various special cases that must be dealt with in order to fully diagonalize the one-loop perturbation and characterize the irrep decomposition. Although the projections onto the various invariant subspaces are matters of simple algebra, that algebra is too complicated to be done by hand and we have resorted to symbolic manipulation programs. The end result of the diagonalization is quite simple and the reader willing to accept our results on faith can skip ahead to the end of this subsection.

We begin with a discussion of the action of the purely bosonic perturbation $\mathcal{H}_{\rm BB}$ on the 64-dimensional Fock space created by pairs of bosonic creation operators. Part of this subspace connects via $\mathcal{H}_{\rm BF}$ to the Fock space of spacetime bosons created by pairs of fermionic creation operators, and we will deal with it later. There is, however, a subspace that only connects to itself, through the purely bosonic perturbation $\mathcal{H}_{\rm BB}$. We will first deal with this purely bosonic block diagonalization, leading to eigenvalues we will denote by $\Lambda_{\rm BB}$. The 8 bosonic modes lie in the $SO(4)\times SO(4)$ representations $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ (i.e. they are vectors in the $SO(4)$ subgroups descended from AdS_5 and S^5, respectively). The key fact about $\mathcal{H}_{\rm BB}$ is that the 16-dimensional spaces spanned by two $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ oscillators or by two $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ oscillators are closed under its action (it is also true that $\mathcal{H}_{\rm BF}$ annihilates both of these subspaces). The $SO(4)$ representation content of the states created by such oscillator pairs is given by the formula $(\mathbf{2},\mathbf{2})\times(\mathbf{2},\mathbf{2}) = (\mathbf{3},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1})$ (we use $SU(2)\times SU(2)$ notation, rather than $SO(4)$, since it is unavoidable when we discuss fermions). By projecting the $\mathcal{O}(\lambda')$ part of (6.6) onto these subspaces, one can directly read off the eigenvalues $\Lambda_{\rm BB}$, with the results shown in Table 2. The identification of the representations associated with particular eigenvalues is easy to do on the basis of multiplicity. In any event, projection onto invariant subspaces is a simple matter of symmetrization or antisymmetrization of oscillator indices and can be done directly. The most important point to note is that the eigenvalues are successive even integers, a simple result and one which is consistent with our expectations from extended supersymmetry. It will be straightforward to match these states to gauge theory operators and compare eigenvalues with anomalous dimensions.

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2
$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0

Table 2: Energy shifts at $\mathcal{O}(1/J)$ for unmixed bosonic modes

The Fock space of spacetime bosons created by pairs of fermionic creation operators contains a similar pair of 16×16 diagonal sub-blocks. The construction and application of the relevant projection operators and the subsequent match-up with gauge theory operators is more complicated than on the bosonic side and we must develop some technical tools before we can obtain concrete results.

Just as $\mathcal{H}_{\rm BB}$ is closed in the two 16-dimensional spaces of bosonic $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ or $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ states, $\mathcal{H}_{\rm FF}$ is closed on subspaces of bi-fermions spanned by a pair of $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or a pair

of $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ fermionic oscillators (i.e. $--$ or $++$ states, to use an obvious shorthand). The complete spectrum of eigenvalues from these subsectors of the Hamiltonian can be computed by projecting out the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ spinors in $\mathcal{H}_{\rm FF}$ (6.7). To do this, it will be helpful to express the 8-component spinors of the string theory in a basis which allows us to define fermionic oscillators labeled by their $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ representation content.

The original 32-component Majorana-Weyl spinors θ^I were reduced by the Weyl projection and a light-cone gauge condition to an 8-component spinor ψ^α (transforming in the 8_s of $SO(8)$). The generators of the four $SU(2)$ factors (6.9) of the manifest $SO(4)\times SO(4)$ symmetry can be expressed as 8×8 $SO(8)$ matrices as follows:

$$
\begin{aligned}
\Sigma_1^\pm &= -\frac{1}{4i}(\gamma^2\gamma^3 \pm \gamma^1\gamma^4) & \Omega_1^\pm &= \frac{1}{4i}(-\gamma^6\gamma^7 \pm \gamma^5)\\
\Sigma_2^\pm &= -\frac{1}{4i}(\gamma^3\gamma^1 \pm \gamma^2\gamma^4) & \Omega_2^\pm &= \frac{1}{4i}(-\gamma^7\gamma^5 \pm \gamma^6)\\
\Sigma_3^\pm &= -\frac{1}{4i}(\gamma^1\gamma^2 \pm \gamma^3\gamma^4) & \Omega_3^\pm &= \frac{1}{4i}(-\gamma^5\gamma^6 \pm \gamma^7)\ .
\end{aligned}
\tag{6.17}
$$

We will use the representation for the γ^A given in the Appendix (A.15) when we need to make these generators explicit. The 8_s spinor may be further divided into its $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ components $\hat\psi$ and $\tilde\psi$, respectively, and this suggests a useful basis change for the string creation operators: for the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ spinor, we define four new objects w,x,y,z by

$$
\hat b^\dagger = w\begin{pmatrix}1\\0\\0\\-1\\0\\0\\0\\0\end{pmatrix} + x\begin{pmatrix}0\\1\\1\\0\\0\\0\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\1\end{pmatrix} + z\begin{pmatrix}0\\1\\1\\0\\0\\1\\-1\\0\end{pmatrix}\ ,
\tag{6.18}
$$

which we then organize in two different ways into two-component complex spinors:

$$
\begin{aligned}
\zeta &= \begin{pmatrix} w+iy\\ z+ix\end{pmatrix} & \varphi &= \begin{pmatrix} -z+ix\\ w-iy\end{pmatrix} & &\Leftarrow\ \Sigma_i^-\\
\bar\zeta &= \begin{pmatrix} w+iy\\ -z+ix\end{pmatrix} & \bar\varphi &= \begin{pmatrix} z+ix\\ w-iy\end{pmatrix} & &\Leftarrow\ \Omega_i^-\ .
\end{aligned}
\tag{6.19}
$$

This organization into 2-spinors is meant to show how components of $\hat\psi$ transform under the two $SU(2)$ factors which act non-trivially on them. As may be verified from the explicit forms of the $SU(2)$ generators obtained by substituting (A.15) into (6.17), the two-component

spinors ζ and φ transform as $(\mathbf{1},\mathbf{2})$ under the first $SO(4)$ and the spinors $\bar{\zeta}$ and $\bar{\varphi}$ transform as $(\mathbf{1},\mathbf{2})$ under the second $SO(4)$ of $SO(4)\times SO(4)$. The explicit realization of the two $SU(2)$ factors involved here is found in this way to be

$$\begin{aligned}
\Sigma_1^- &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} & \Omega_1^- &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \\
\Sigma_2^- &= \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix} & \Omega_2^- &= \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \\
\Sigma_3^- &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & \Omega_3^- &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} .
\end{aligned} \tag{6.20}$$

One may similarly decompose $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ spinors and express the corresponding generators Σ^+ and Ω^+. We decompose $\tilde{\psi}$ into components $\bar{w},\bar{x},\bar{y},\bar{z}$ according to

$$\tilde{b}^\dagger = \bar{w}\begin{pmatrix} 1\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix} + \bar{x}\begin{pmatrix} 0\\1\\-1\\0\\0\\0\\0\\0 \end{pmatrix} + \bar{y}\begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\-1 \end{pmatrix} + \bar{z}\begin{pmatrix} 0\\0\\0\\0\\0\\1\\1\\0 \end{pmatrix} , \tag{6.21}$$

and rearrange them into two-component complex spinors:

$$\begin{aligned}
\xi &= \begin{pmatrix} \bar{z}+i\bar{x} \\ \bar{w}+i\bar{y} \end{pmatrix} & \eta &= \begin{pmatrix} \bar{w}-i\bar{y} \\ -\bar{z}+i\bar{x} \end{pmatrix} & &\Leftarrow \quad \Sigma_i^+ \\
\bar{\xi} &= \begin{pmatrix} -\bar{z}+i\bar{x} \\ \bar{w}+i\bar{y} \end{pmatrix} & \bar{\eta} &= \begin{pmatrix} \bar{w}-i\bar{y} \\ \bar{z}+i\bar{x} \end{pmatrix} & &\Leftarrow \quad \Omega_i^+ .
\end{aligned} \tag{6.22}$$

The corresponding explicit $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ generators are given by

$$\begin{aligned}
\Sigma_1^+ &= \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} & \Omega_1^+ &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \\
\Sigma_2^+ &= \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix} & \Omega_2^+ &= \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \\
\Sigma_3^+ &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & \Omega_3^+ &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} .
\end{aligned} \tag{6.23}$$

These observations will make it possible to construct linear combinations of products of components of ψ^α transforming in chosen irreps of $SO(4) \times SO(4)$.

Let us now use this machinery to analyze the perturbation matrix on spacetime bosons created by two fermionic creation operators (bi-fermions). As explained in the discussion of (6.7), $\mathcal{H}_{\rm FF}$ is block-diagonal on the 16-dimensional $++$ or $--$ bi-fermionic subspaces. To project out the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $++$ block of $\mathcal{H}_{\rm FF}$, we simply act on all indices of (6.7) with the Π_+ projection operator:

$$
\begin{aligned}
\left\langle J \middle| \tilde{b}^\alpha_n \tilde{b}^\beta_{-n} \left(\mathcal{H}_{\rm FF}\right) \tilde{b}^{\gamma\dagger}_{-n} \tilde{b}^{\delta\dagger}_n \middle| J \right\rangle \;=\; & -2\frac{n^2\lambda'}{J}\Pi_+^{\alpha\delta}\Pi_+^{\beta\gamma} + \frac{n^2\lambda'}{24J}\Bigg\{\Bigg[(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\delta}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\gamma} \\
& +(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{ij}\Pi_+)^{\gamma\delta} - (\Pi_+\gamma^{ij}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\delta}\Bigg] \\
& -\Bigg[(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\delta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\gamma} + (\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\gamma\delta} \\
& -(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\delta}\Bigg]\Bigg\}\ . \qquad (6.24)
\end{aligned}
$$

The $SO(4) \times SO(4)$ representation content of this subspace is specified by $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) \times (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) = (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$ and we must further project onto individual irreducible representations in order to identify the eigenvalues.

With the tools we have built up in the last few paragraphs, we are in a position to directly project out some of the desired irreducible representations. Bi-fermions of $++$ type transforming as scalars under the first $SO(4)$ (i.e. under Σ_i^+) are constructed by making $SU(2)$ invariants out of the two-component spinors ξ and η. There are four such objects:

$$
\begin{array}{ll}
\xi_{-n}\tau_2\xi_n & \qquad \xi_{-n}\tau_2\eta_n \\
\eta_{-n}\tau_2\xi_n & \qquad \eta_{-n}\tau_2\eta_n\ ,
\end{array}
\qquad (6.25)
$$

where τ_2 is the second Pauli matrix. At the same time, they must also comprise a $\mathbf{3}$ and a $\mathbf{1}$ under the second $SO(4)$ (i.e. under Ω_i^+). To identify the irreducible linear combinations, one has to re-express the objects in (6.25) in terms of the spinors $\bar{\xi}$ and $\bar{\eta}$ that transform simply under Ω_i^+. Representative results for properly normalized creation operators of $++$ bi-fermion states in particular $SO(4) \times SO(4)$ irreps are

$$
\begin{array}{lll}
-\frac{1}{2}\left(\xi_{-n}\tau_2\eta_n - \eta_{-n}\tau_2\xi_n\right) & \qquad (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) & \Lambda_{\rm FF} = -2 \\[2ex]
\left.\begin{array}{l}
\frac{1}{2}\left(\xi_{-n}\tau_2\eta_n + \eta_{-n}\tau_2\xi_n\right) \\
\frac{i}{2}\left(\xi_{-n}\tau_2\xi_n + \eta_{-n}\tau_2\eta_n\right) \\
-\frac{1}{2}\left(\xi_{-n}\tau_2\xi_n - \eta_{-n}\tau_2\eta_n\right)
\end{array}\right\} & \qquad (\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) & \Lambda_{\rm FF} = 0\ .
\end{array}
\qquad (6.26)
$$

We simply have to re-express the ξ, η bilinears in terms of the original spinor creation operators $\tilde{b}$ in order to obtain an explicit projection of the matrix elements (6.24) onto irreducible

subspaces and to obtain the eigenvalues Λ_{FF} associated with each irrep. A parallel analysis of states constructed by forming normalized $SU(2)$ invariants from $\bar{\xi}$ and $\bar{\eta}$ gives another irrep and eigenvalue:

$$\left.\begin{array}{l}\frac{1}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\eta}_n+\bar{\eta}_{-n}\tau_2\bar{\xi}_n\right)\\ \frac{i}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\xi}_n+\bar{\eta}_{-n}\tau_2\bar{\eta}_n\right)\\ -\frac{1}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\xi}_n-\bar{\eta}_{-n}\tau_2\bar{\eta}_n\right)\end{array}\right\}(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})\qquad \Lambda_{\rm FF}=-4\ . \tag{6.27}$$

By similar arguments, whose details we will omit, one can construct the creation operator for the normalized $(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$ or $++$ bi-fermion and find the eigenvalue $\Lambda_{FF}=-2$.

An exactly parallel analysis of $\left\langle J\middle|\hat{b}\hat{b}(\mathcal{H}_{\rm FF})\hat{b}^\dagger\hat{b}^\dagger\middle|J\right\rangle$ on the 16-dimensional subspace spanned by $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ bi-fermions yields the same eigenvalue spectrum. The creation operators of irreducible states (built this time out of ζ and ϕ) and their eigenvalues are

$$-\tfrac{1}{2}\left(\zeta_{-n}\tau_2\varphi_n-\varphi_{-n}\tau_2\zeta_n\right)\qquad(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})\qquad \Lambda_{\rm FF}=-2$$

$$\left.\begin{array}{l}\frac{1}{2}\left(\zeta_{-n}\tau_2\varphi_n+\varphi_{-n}\tau_2\zeta_n\right)\\ \frac{i}{2}\left(\zeta_{-n}\tau_2\zeta_n+\varphi_{-n}\tau_2\varphi_n\right)\\ -\frac{1}{2}\left(\zeta_{-n}\tau_2\zeta_n-\varphi_{-n}\tau_2\varphi_n\right)\end{array}\right\}(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})\qquad \Lambda_{\rm FF}=0 \tag{6.28}$$

$$\left.\begin{array}{l}\frac{1}{2}\left(\bar{\zeta}_{-n}\tau_2\bar{\varphi}_n+\bar{\varphi}_{-n}\tau_2\bar{\zeta}_n\right)\\ \frac{i}{2}\left(\bar{\zeta}_{-n}\tau_2\bar{\zeta}_n+\bar{\varphi}_{-n}\tau_2\bar{\varphi}_n\right)\\ -\frac{1}{2}\left(\bar{\zeta}_{-n}\tau_2\bar{\zeta}_n-\bar{\varphi}_{-n}\tau_2\bar{\varphi}_n\right)\end{array}\right\}(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})\qquad \Lambda_{\rm FF}=-4\ . \tag{6.29}$$

The overall results for this sector are displayed in Table 3.

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})$	0
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$	-2

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2

Table 3: Energy shifts of states created by two fermions in $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$

To this point, we have been able to study specific projections of the $\mathcal{H}_{\rm BB}$ and $\mathcal{H}_{\rm FF}$ subsectors by choosing states that are not mixed by $\mathcal{H}_{\rm BF}$. We now must deal with the subspace of spacetime boson two-impurity states that is not annihilated by $\mathcal{H}_{\rm BF}$. This 64-dimensional space is spanned by pairs of bosonic creation operators taken from different $SO(4)$ subgroups and pairs of fermionic creation operators of opposite Π-parity. The representation content of these creation-operator pairs is such that the states in this sector all belong to $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$ irreps. This space is of course also acted on by $\mathcal{H}_{\rm BB}$ and $\mathcal{H}_{\rm FF}$, so we will need the matrix

elements of all three pieces of the Hamiltonian as they act on this subspace. By applying the appropriate projections to the general one-loop matrix elements, we obtain the expressions

$$\left\langle J\middle| a_n^A a_{-n}^B \left(\mathcal{H}_{\rm BB}\right) a_{-n}^{C\dagger} a_n^{D\dagger}\middle| J\right\rangle \to -2\frac{n^2\lambda'}{J}\left(\delta^{ad'}\delta^{b'c}+\delta^{a'd}\delta^{bc'}+\delta^{ad}\delta^{b'c'}+\delta^{a'd'}\delta^{bc}\right) \quad (6.30)$$

$$\begin{aligned}\left\langle J\middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm BF}\right) a_{-n}^{A\dagger} a_n^{B\dagger}\middle| J\right\rangle \quad \to \quad & \frac{n^2\lambda'}{2J}\Big[\left(\Pi_+\gamma^{ab'}\Pi_-\right)^{\alpha\beta}-\left(\Pi_+\gamma^{a'b}\Pi_-\right)^{\alpha\beta}+\left(\Pi_-\gamma^{ab'}\Pi_+\right)^{\alpha\beta} \\ & -\left(\Pi_-\gamma^{a'b}\Pi_+\right)^{\alpha\beta}\Big] \end{aligned} \quad (6.31)$$

$$\begin{aligned}\left\langle J\middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm FF}\right) b_{-n}^{\gamma\dagger} b_n^{\delta\dagger}\middle| J\right\rangle \to & -2\frac{n^2\lambda'}{J}\left(\Pi_+^{\alpha\delta}\Pi_-^{\beta\gamma}+\Pi_-^{\alpha\delta}\Pi_+^{\beta\gamma}\right) \\ & +\frac{n^2\lambda'}{24J}\Big\{\Big[(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\delta}(\Pi_-\gamma^{ij}\Pi_-)^{\beta\gamma}+(\Pi_+\gamma^{ij}\Pi_-)^{\alpha\beta}(\Pi_-\gamma^{ij}\Pi_+)^{\gamma\delta} \\ & -(\Pi_+\gamma^{ij}\Pi_-)^{\alpha\gamma}(\Pi_-\gamma^{ij}\Pi_+)^{\beta\delta}\Big]-\Big[(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\delta}(\Pi_-\gamma^{i'j'}\Pi_-)^{\beta\gamma} \\ & +(\Pi_+\gamma^{i'j'}\Pi_-)^{\alpha\beta}(\Pi_-\gamma^{i'j'}\Pi_+)^{\gamma\delta}-(\Pi_+\gamma^{i'j'}\Pi_-)^{\alpha\gamma}(\Pi_-\gamma^{i'j'}\Pi_+)^{\beta\delta}\Big] \\ & +\Big[(\Pi_-\gamma^{ij}\Pi_-)^{\alpha\delta}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\gamma}+(\Pi_-\gamma^{ij}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{ij}\Pi_-)^{\gamma\delta} \\ & -(\Pi_-\gamma^{ij}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{ij}\Pi_-)^{\beta\delta}\Big]-\Big[(\Pi_-\gamma^{i'j'}\Pi_-)^{\alpha\delta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\gamma} \\ & +(\Pi_-\gamma^{i'j'}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{i'j'}\Pi_-)^{\gamma\delta}-(\Pi_-\gamma^{i'j'}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{i'j'}\Pi_-)^{\beta\delta}\Big]\Big\}\ . \end{aligned} \quad (6.32)$$

Since the 64-dimensional space must contain four copies of the $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$ irrep, the diagonalization problem is really only 4×4 and quite easy to solve. The results for the eigenvalues appear in Table 4. Collecting the above results, we present the complete $SO(4)_{AdS}\times SO(4)_{S^5}$

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm BF}$
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})\times 2$	-2
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0

Table 4: String eigenstates in the subspace for which $\mathcal{H}_{\rm BF}$ has non-zero matrix elements

decomposition of spacetime boson two-impurity states in Table 5.

By projecting out closed subspaces of the one-loop Hamiltonian we have successfully classified each of the energy levels in the bosonic Fock space with an $SO(4)\times SO(4)$ symmetry label. Similar arguments can be applied to the fermionic Fock space, where two-impurity string states mix individual bosonic and fermionic oscillators (we omit the details). A summary of these results for all states, including spacetime fermions, is given in Table 6. The

	$SO(4)_{AdS} \times SO(4)_{S^5}$	Λ
$\mathcal{H}_{\rm BB}$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4
	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2

	$SO(4)_{AdS} \times SO(4)_{S^5}$	Λ
$\mathcal{H}_{\rm FF}$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
	$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2
$\mathcal{H}_{\rm BF}$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0
	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})\times 2$	-2
	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4

Table 5: Group decomposition of the 128 two-impurity spacetime bosons

important fact to note is that the Λ eigenvalues and their multiplicities are exactly as required for consistency with the full $PSU(2,2|4)$ symmetry of the theory. This is a non-trivial result since the quantization procedure does not make the full symmetry manifest. It is also a very satisfying check of the overall correctness of the extremely complicated set of procedures we were forced to use. We can now proceed to a comparison with gauge theory anomalous dimensions.

Level	0	2	4	6	8
Mult.	1	28	70	28	1
$\Lambda_{\rm Bose}$	-6	-4	-2	0	2

Level	1	3	5	7
Mult.	8	56	56	8
$\Lambda_{\rm Fermi}$	-5	-3	-1	1

Table 6: First-order energy shift summary: complete two-impurity string multiplet

6.6 Gauge theory comparisons

The most comprehensive analysis of one-loop anomalous dimensions of BMN operators and their organization into supersymmetry multiplets was given in [18]. As stated in our previous summary publication [4], the above string theory calculations are in perfect agreement with the one-loop gauge theory predictions. For completeness, we present a summary of the spectrum of dimensions of gauge theory operators along with a sampling of information about their group transformation properties.

The one-loop formula for operator dimensions takes the generic form

$$\Delta_n^{\mathcal{R}} = 2 + \frac{g_{YM}^2 N_c}{\mathcal{R}^2} n^2 \left(1 + \frac{\bar{\Lambda}}{\mathcal{R}} + \mathcal{O}(\mathcal{R}^{-2})\right) . \tag{6.33}$$

The $\mathcal{O}(\mathcal{R}^{-1})$ correction $\bar{\Lambda}$ for the set of two-impurity operators is predicted to match the corresponding $\mathcal{O}(J^{-1})$ energy correction to two-impurity string states, labeled above by Λ.

Part of the motivation for performing the special projections on two-impurity string states detailed above was to emerge with specific symmetry labels for each of the string eigenstates. String states of a certain representation content of the residual $SO(4) \times SO(4)$ symmetry of $AdS_5 \times S^5$ are expected, by duality, to map to gauge theory operators with the same representation labels in the $SL(2, \mathbf{C})$ Lorentz and $SU(4)$ $\mathcal{R}$-charge sectors of the gauge theory. Knowing the symmetry content of the string eigenstates therefore allows us to test this mapping in detail.

The bosonic sector of the gauge theory, characterized by single-trace operators with two bosonic insertions in the trace, appears in Table 7. The set of operators comprising Lorentz

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\Sigma_A \operatorname{tr}\left(\phi^A Z^p \phi^A Z^{R-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	-6
$\operatorname{tr}\left(\phi^{(i} Z^p \phi^{j)} Z^{R-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{3}, \mathbf{3})$	-2
$\operatorname{tr}\left(\phi^{[i} Z^p \phi^{j]} Z^{R-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{3})$	-4
$\operatorname{tr}\left(\nabla_\mu Z Z^p \nabla^\mu Z Z^{R-2-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	2
$\operatorname{tr}\left(\nabla_{(\mu} Z Z^p \nabla_{\nu)} Z Z^{R-2-p}\right)$	$(\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$	-2
$\operatorname{tr}\left(\nabla_{[\mu} Z Z^p \nabla_{\nu]} Z Z^{R-2-p}\right)$	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}; \mathbf{1}, \mathbf{1})$	0

Table 7: Bosonic gauge theory operators: either spacetime or $\mathcal{R}$-charge singlet.

scalars clearly agree with the corresponding pure-boson string states in Table 5 which are scalars in AdS_5. Operators containing pairs of spacetime derivatives correspond to string theory states that are scalars of the S^5 subspace. The bi-fermion sector of the string theory corresponds to the set of two-gluino operators in the gauge theory. A few of these operators are listed in Table 8. These states, which form either spacetime or $\mathcal{R}$-charge scalars, clearly

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\operatorname{tr}\left(\chi^{[\alpha} Z^p \chi^{\beta]} Z^{R-1-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	-2
$\operatorname{tr}\left(\chi^{(\alpha} Z^p \chi^{\beta)} Z^{R-1-p}\right)$	$(\mathbf{1}, \mathbf{1}; \mathbf{3}, \mathbf{1})$	0
$\operatorname{tr}\left(\chi[\sigma_\mu, \tilde{\sigma}_\nu] Z^p \chi Z^{R-1-p}\right)$	$(\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1})$	-4

Table 8: Bosonic gauge theory operators with two gluino impurities.

agree with their string theory counterparts which were constructed explicitly above. The string states appearing in the $(\mathbf{2}, \mathbf{2}; \mathbf{2}, \mathbf{2})$ representation (listed in Table 4) correspond to the operators listed in Table 9. Finally, the complete supermultiplet spectrum of two-impurity gauge theory operators appears in Table 10. The extended supermultiplet spectrum is in perfect agreement with the complete one-loop string theory spectrum in Table 6 above.

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right) + \ldots$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right)$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-2
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right) + \ldots$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0

Table 9: Bosonic gauge theory operators: spacetime and $\mathcal{R}$-charge non-singlets

Level	0	1	2	3	4	5	6	7	8
Multiplicity	1	8	28	56	70	56	28	8	1
$\delta E \times (\mathcal{R}^2/g_{YM}^2 N_c n^2)$	$-6/\mathcal{R}$	$-5/\mathcal{R}$	$-4/\mathcal{R}$	$-3/\mathcal{R}$	$-2/\mathcal{R}$	$-1/\mathcal{R}$	0	$1/\mathcal{R}$	$2/\mathcal{R}$

Table 10: Anomalous dimensions of two-impurity operators

7 Energy spectrum at all loops in λ'

To make comparisons with gauge theory dimensions at one loop in $\lambda = g_{YM}^2 N_c$, we have expanded all string energies in powers of the modified 't Hooft coupling $\lambda' = g_{YM}^2 N_c/\mathcal{R}^2$. The string theory analysis is exact to all orders in λ', however, and it is possible to extract a formula for the $\mathcal{O}(1/J)$ string energy corrections which is exact in λ' and suitable for comparison with higher-order corrections to operator dimensions in the gauge theory. In practice, it is slightly more difficult to diagonalize the string Hamiltonian when the matrix elements are not expanded in small λ'. This is mainly because, beyond leading order, $\mathcal{H}_{\rm BF}$ acquires additional terms that mix bosonic indices in the same $SO(4)$ and also mix bi-fermionic indices in the same $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ representation. Instead of a direct diagonalization of the entire 128-dimensional subspace of spacetime bosons, for example, we find it more convenient to exploit the 'dimension reduction' that can be achieved by projecting the full Hamiltonian onto individual irreps.

For example, the $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ irrep appears four times in Table 5 and is present at levels $L = 0, 4, 8$ in the supermultiplet. To get the exact eigenvalues for this irrep, we will have to diagonalize a 4×4 matrix. The basis vectors of this bosonic sector comprise singlets of the two $SO(4)$ subgroups ($a^{\dagger a} a^{\dagger a}\,|J\rangle$ and $a^{\dagger a'} a^{\dagger a'}\,|J\rangle$) plus two bi-fermion singlets constructed from the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ creation operators ($\hat{b}^{\dagger\alpha}\hat{b}^{\dagger\alpha}\,|J\rangle$ and $\tilde{b}^{\dagger\alpha}\tilde{b}^{\dagger\alpha}\,|J\rangle$). The different Hamiltonian matrix elements that enter the 4×4 matrix are symbolically indicated in Table 11. It is a simple matter to project the general expressions for matrix elements of $\mathcal{H}_{\rm BB}$, etc., onto singlet states and so obtain the matrix as an explicit function of λ', n. The

matrix can be exactly diagonalized and yields the following energies:

$$\begin{aligned}
E_0(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2) \\
E_4(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{2n^2\lambda'}{J} + \mathcal{O}(1/J^2) \\
E_8(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2-\frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\ .
\end{aligned} \tag{7.1}$$

The subscript $L = 0, 4, 8$ indicates the supermultiplet level to which the eigenvalue connects in the weak coupling limit. The middle eigenvalue (L=4) is doubly degenerate, as it was in the one-loop limit.

	$a^{\dagger a}a^{\dagger a}\,\|J\rangle$	$a^{\dagger a'}a^{\dagger a'}\,\|J\rangle$	$\hat{b}^{\dagger\alpha}\hat{b}^{\dagger\alpha}\,\|J\rangle$	$\tilde{b}^{\dagger\alpha}\tilde{b}^{\dagger\alpha}\,\|J\rangle$
$\langle J\|\, a^a a^a$	$\mathcal{H}_{\mathrm{BB}}$	$\mathcal{H}_{\mathrm{BB}}$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{BF}}$
$\langle J\|\, a^{a'} a^{a'}$	$\mathcal{H}_{\mathrm{BB}}$	$\mathcal{H}_{\mathrm{BB}}$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{BF}}$
$\langle J\|\, \hat{b}^\alpha \hat{b}^\alpha$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{FF}}$	$\mathcal{H}_{\mathrm{FF}}$
$\langle J\|\, \tilde{b}^\alpha \tilde{b}^\alpha$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{BF}}$	$\mathcal{H}_{\mathrm{FF}}$	$\mathcal{H}_{\mathrm{FF}}$

Table 11: Singlet projection at finite λ'

There are two independent 2×2 matrices that mix states at levels $L = 2, 6$. According to Table 5, one can project out the antisymmetric bosonic and antisymmetric bi-fermionic states in the irrep $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$ or in the irrep $(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$. The results of eqns. (6.26,6.27, 6.28,6.29) can be used to carry out the needed projections and obtain explicit forms for the matrix elements of the perturbing Hamiltonian. The actual 2×2 diagonalization is trivial to do and both problems give the same result. The final result for the energy levels (using the same notation as before) is

$$\begin{aligned}
E_2(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2) \\
E_6(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2-\frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\ .
\end{aligned} \tag{7.2}$$

We can carry out similar diagonalizations for the remaining irreps of Table 5, but no new eigenvalues are encountered: the energies already listed are the exact energies of the $L = 0, 2, 4, 6, 8$ levels. It is also easy to see that the degeneracy structure of the exact levels is the same as the one-loop degeneracy.

The odd levels of the supermultiplet are populated by the 128-dimensional spacetime fermions, and this sector of the theory can be diagonalized directly. Proceeding in a similar fashion as in the bosonic sector, we find exact energy eigenvalues for the $L = 1, 3, 5, 7$ levels

(with unchanged multiplicities). We refrain from stating the individual results because the entire supermultiplet spectrum, bosonic and fermionic, can be written in terms of a single concise formula: to leading order in $1/J$ and all orders in λ', the energies of the two-impurity multiplet are given by

$$E_L(n,J) \;=\; 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{(4-L)}{\sqrt{1+n^2\lambda'}}\right] + O(1/J^2)\ , \tag{7.3}$$

where $L = 0, 1, \ldots, 8$ indicates the level within the supermultiplet. The degeneracies and irrep content are identical to what we found at one loop in λ'. This expression can be rewritten, correct to order J^{-2}, as follows:

$$E_L(n,J) \simeq 2\sqrt{1+\frac{\lambda n^2}{(J-L/2)^2}} - \frac{n^2\lambda}{(J-L/2)^3}\left[2+\frac{4}{\sqrt{1+\lambda n^2/(J-L/2)^2}}\right]\ . \tag{7.4}$$

This shows that, within this expansion, the joint dependence on J and L is exactly what is required for extended supersymmetry multiplets. This is a rather nontrivial functional requirement, and a stringent check on the correctness of our quantization procedure (independent of any comparison with gauge theory).

In order to make contact with gauge theory we expand (7.3) in λ', obtaining

$$\begin{aligned} E_L(n,J) \;\approx\; & \left[2+\lambda' n^2 - \frac{1}{4}(\lambda' n^2)^2 + \frac{1}{8}(\lambda' n^2)^3 + \ldots\right] \\ & + \frac{1}{J}\left[n^2\lambda'(L-6) + (n^2\lambda')^2\left(\frac{4-L}{2}\right) + (n^2\lambda')^3\left(\frac{3L-12}{8}\right) + \ldots\right]\ . \end{aligned} \tag{7.5}$$

We can now address the comparison with higher-loop results on gauge theory operator dimensions. Beisert, Kristjansen and Staudacher [19] computed the two-loop correction to the anomalous dimensions of a convenient class of operators lying at level four in the supermultiplet. The operators in question lie in a symmetric-traceless irrep of an $SO(4)$ subgroup of the $\mathcal{R}$-charge and are guaranteed by group theory not to mix with any other fields [19]. The following expression for the two-loop anomalous dimension was found:

$$\delta\Delta_n^{\mathcal{R}} \;=\; -\frac{g_{YM}^4 N_c^2}{\pi^4}\sin^4\frac{n\pi}{\mathcal{R}+1}\left(\frac{1}{4}+\frac{\cos^2\frac{n\pi}{\mathcal{R}+1}}{\mathcal{R}+1}\right)\ . \tag{7.6}$$

As explained above, $\mathcal{N}=4$ supersymmetry insures that the dimensions of operators at other levels of the supermultiplet will be obtained by making the substitution $\mathcal{R}\to\mathcal{R}+2-L/2$ in the expression for the dimension of the $L=4$ operator. Making that substitution and taking the large-$\mathcal{R}$ limit we obtain a general formula for the two-loop, large-$\mathcal{R}$ correction to the anomalous dimension of the general two-impurity operator:

$$\begin{aligned} \delta\Delta_n^{\mathcal{R},L} \;=\; & -\frac{g_{YM}^4 N_c^2}{\pi^4}\sin^4\frac{n\pi}{\mathcal{R}+3-L/2}\left(\frac{1}{4}+\frac{\cos^2\frac{n\pi}{\mathcal{R}+3-L/2}}{\mathcal{R}+3-L/2}\right) \\ \approx\; & -\frac{1}{4}(\lambda' n^2)^2 + \frac{1}{2}(\lambda' n^2)^2\,\frac{4-L}{\mathcal{R}} + O(1/\mathcal{R}^2)\ , \end{aligned} \tag{7.7}$$

Using the identification $\mathcal{R} \rightleftharpoons J$ specified by duality, we see that this expression matches the corresponding string result in (7.5) to $\mathcal{O}(1/J)$, confirming the AdS/CFT correspondence to two loops in the gauge coupling.

The three-loop correction to the dimension of this same class of $L = 4$ gauge theory operators has recently been definitively determined [20]. The calculation involves a remarkable interplay between gauge theory and integrable spin chain models [5, 19, 21, 22]. The final result is

$$\delta\Delta_n^{\mathcal{R}} \;=\; \left(\frac{\lambda}{\pi^2}\right)^3 \sin^6 \frac{n\pi}{\mathcal{R}+1}\left[\frac{1}{8} + \frac{\cos^2 \frac{n\pi}{\mathcal{R}+1}}{4(\mathcal{R}+1)^2}\left(3\mathcal{R} + 2(\mathcal{R}+6)\cos^2 \frac{n\pi}{\mathcal{R}+1}\right)\right] . \quad (7.8)$$

If we apply to this expression the same logic applied to the two-loop gauge theory result (7.6), we obtain the following three-loop correction to the anomalous dimension of the general level of the two-impurity operator supermultiplet:

$$\delta\Delta_n^{\mathcal{R},L} \approx \frac{1}{8}(\lambda' n^2)^3 - \frac{1}{8}(\lambda' n^2)^2\,\frac{8-3L}{\mathcal{R}} + O(1/\mathcal{R}^2) . \quad (7.9)$$

We see that this expression differs from the third-order contribution to the string result (7.5) for the corresponding quantity. The difference is a constant shift and one might hope to absorb it in a normal-ordering constant. However, our discussion of the normal-ordering issue earlier in the paper seems to exclude any such freedom.

8 Discussion and conclusions

As a complement to the work presented in [4], we have given a detailed account of the quantization of the first curvature correction to type IIB superstring theory in the plane-wave limit of $AdS_5 \times S^5$. We have presented the detailed diagonalization of the resulting perturbing Hamiltonian on the degenerate subspace of two-impurity states, obtaining string energy corrections that can be compared with higher-loop anomalous dimensions of gauge theory operators. Beyond the Penrose limit, the holographic mapping between each side of the correspondence is intricate and nontrivial, and works perfectly to two loops in the gauge coupling. The agreement, however, appears to break down at three loops. (A similar three-loop disagreement appeared more recently in a semiclassical string analysis presented in [23].) This troubling issue was first observed in [4], at which time the third-order gauge theory anomalous dimension was somewhat conjectural. In the intervening time, the third-order result (7.8) has acquired a solid basis, thus confirming the mismatch. Several questions arise about this mismatch: is it due to a failure of the AdS/CFT correspondence itself, does it signal the need to modify the worldsheet string action, or is it simply that the perturbative approach to the gauge theory anomalous dimensions is not adequate in the relevant limits? Despite vigorous investigation from several directions, all these questions remain open. One very promising development has been the recognition of integrable structures on both sides of the duality. They have been the focus of many recent studies, and one may hope that

integrability will be a guide to recovering the correspondence beyond two loops or at least achieving some positive understanding of the mismatch. Certainly the final solution will augment our understanding of integrability and of the AdS/CFT mechanism in general.

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A Notation and conventions

The various indices are chosen to represent the following:

$$\begin{array}{rl}
\mu,\nu,\rho = 0,\ldots,9 & SO(9,1) \text{ vectors} \\
\alpha,\beta,\gamma,\delta = 1,\ldots,16 & SO(9,1) \text{ spinors} \\
A,B = 1,\ldots,8 & SO(8) \text{ vectors} \\
i,j,k = 1,\ldots,4 & SO(4) \text{ vectors} \\
i',j',k' = 5,\ldots,8 & SO(4)' \text{ vectors} \\
a,b = 0,1 & \text{worldsheet coordinates } (\tau,\sigma) \\
I,J,K,L = 1,2 & \text{label two MW spinors of equal chirality.}
\end{array} \tag{A.1}$$

The 32×32 Dirac gamma matrices are decomposed into a 16×16 representation according to

$$\begin{array}{rl}
(\Gamma^\mu)_{32\times32} = \begin{pmatrix} 0 & \gamma^\mu \\ \bar\gamma^\mu & 0 \end{pmatrix} & \gamma^\mu\bar\gamma^\nu + \gamma^\nu\bar\gamma^\mu = 2\eta^{\mu\nu} \\
\gamma^\mu = (1,\gamma^A,\gamma^9) & \bar\gamma^\mu = (-1,\gamma^A,\gamma^9) \\
\gamma^+ = 1+\gamma^9 & \bar\gamma^+ = -1+\gamma^9\ .
\end{array} \tag{A.2}$$

In particular, the notation $\bar\gamma^\mu$ lowers the $SO(9,1)$ spinor indices α,β:

$$\gamma^\mu = (\gamma^\mu)^{\alpha\beta} \qquad \bar\gamma^\mu = (\gamma^\mu)_{\alpha\beta}\ . \tag{A.3}$$

These conventions are chosen to match those of Metsaev in [2]. By invoking κ-symmetry,

$$\bar\gamma^+\theta = 0 \implies \bar\gamma^9\theta = \theta \tag{A.4}$$

$$\bar\gamma^- = 1+\bar\gamma^9 \implies \bar\gamma^-\theta = 2\theta\ . \tag{A.5}$$

The antisymmetric product $\gamma^{\mu\nu}$ is given by

$$\begin{aligned}(\gamma^{\mu\nu})^\alpha{}_\beta &\equiv \frac{1}{2}(\gamma^\mu\bar{\gamma}^\nu)^\alpha{}_\beta - (\mu \rightleftharpoons \nu)\\ (\bar{\gamma}^{\mu\nu})^\alpha{}_\beta &\equiv \frac{1}{2}(\bar{\gamma}^\mu\gamma^\nu)_\alpha{}^\beta - (\mu \rightleftharpoons \nu)\ .\end{aligned} \tag{A.6}$$

We form the matrices Π and $\tilde{\Pi}$ according to:

$$\begin{aligned}\Pi &\equiv \gamma^1\bar{\gamma}^2\gamma^3\bar{\gamma}^4\\ \tilde{\Pi} &\equiv \gamma^5\bar{\gamma}^6\gamma^7\bar{\gamma}^8\ .\end{aligned} \tag{A.7}$$

These form the projection operators ($\Pi^2 = \tilde{\Pi}^2 = 1$)

$$\begin{aligned}\Pi_+ &\equiv \frac{1}{2}(1+\Pi) \qquad & \Pi_- &\equiv \frac{1}{2}(1-\Pi)\\ \tilde{\Pi}_+ &\equiv \frac{1}{2}(1+\tilde{\Pi}) \qquad & \tilde{\Pi}_- &\equiv \frac{1}{2}(1-\tilde{\Pi})\ .\end{aligned} \tag{A.8}$$

The spinors θ^I represent two 32-component Majorana-Weyl spinors of $SO(9,1)$ with equal chirality. The 32-component Weyl condition is $\Gamma_{11}\theta = \theta$, with

$$\Gamma_{11} = \Gamma^0 \ldots \Gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{32\times 32}\ . \tag{A.9}$$

The Weyl condition is used to select the top 16 components of θ to form the 16-component spinors

$$\theta^I \;=\; \begin{pmatrix} \theta^\alpha \\ 0 \end{pmatrix}^I\ . \tag{A.10}$$

It is useful to form a single complex 16-component spinor ψ from the real spinors θ^1 and θ^2:

$$\psi \;=\; \sqrt{2}(\theta^1 + i\theta^2)\ . \tag{A.11}$$

The 16-component Weyl condition $\gamma^9\theta = \theta$ selects the upper 8 components of θ, with

$$\gamma^9 = \gamma^1 \ldots \gamma^8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{16\times 16}\ . \tag{A.12}$$

The 16-component Dirac matrices γ^μ can, in turn, be constructed from the familiar Spin(8) Clifford algebra, wherein (in terms of $SO(8)$ vector indices)

$$(\gamma^A)_{16\times 16} = \begin{pmatrix} 0 & \gamma^A \\ (\gamma^A)^T & 0 \end{pmatrix}\ , \tag{A.13}$$

and

$$\{\gamma^A, \gamma^B\}_{16\times 16} = 2\delta^{AB} \qquad \left(\gamma^A(\gamma^B)^T + \gamma^B(\gamma^A)^T = 2\delta^{AB}\right)_{8\times 8} \ . \tag{A.14}$$

The Spin(8) Clifford algebra may be constructed explicitly in terms of 8 real matrices

$$\begin{aligned} \gamma^1 &= \epsilon \times \epsilon \times \epsilon & \gamma^5 &= \tau_3 \times \epsilon \times 1 \\ \gamma^2 &= 1 \times \tau_1 \times \epsilon & \gamma^6 &= \epsilon \times 1 \times \tau_1 \\ \gamma^3 &= 1 \times \tau_3 \times \epsilon & \gamma^7 &= \epsilon \times 1 \times \tau_3 \\ \gamma^4 &= \tau_1 \times \epsilon \times 1 & \gamma^8 &= 1 \times 1 \times 1 \ , \end{aligned} \tag{A.15}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{A.16}$$

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